Physics 221B
Spring 2011
Final Exam
Begin Time: 9am, Monday, May 9, 2011
Exams must be placed in the homework box in
251 LeConte by 6pm, Tuesday, May 10, 2011

This exam has 100 points total.

In this exam, you may refer to the textbooks (both Sakurai books from both semesters), to the Notes and handouts from both semesters (for example, the xeroxed chapters from Bjorken and Drell), to the homeworks and homework solutions, to your own class notes, and to Jackson's book on eletricity and magnetism (a reference on special relativity). You may not use any other references, nor may you talk to anyone about the exam.

If you have questions I will hold office hours Tuesday, 9-10am and 1-2pm. You can also send me an email but I will be out of town on Monday and unable to check my email, and I don't guarantee that I can always make an instant response to your emails even on Tuesday. I'll do my best, however.

If questions of special relativity arise, I will follow the conventions of Jackson (not Bjorken and Drell).

1. (JJ-coupling; 30 points) The Hamiltonian (in atomic units) for a neutral atom ($N = Z$) can be written $H = H_0 + H_2 + H_1$, where

$$H_0 = \sum_{i=1}^{Z} \Big(\frac{p_i^2}{2} - \frac{Z}{r_i} + \bar{V}_i \Big),$$

$$H_2 = \sum_{i=1}^{Z} \xi(r_i) \mathbf{L}_i \cdot \mathbf{S}_i,$$

$$H_1 = \sum_{i=1}^{Z} (-\bar{V}_i) + \sum_{i<j} \frac{1}{r_{ij}}. \tag{1.1}$$

Here $\bar{V}_i$ is the rotationally invariant, self-consistent potential determined by Hartree-Fock methods, which consists of a local plus a nonlocal term; $\xi(r)$ gives the radial dependence of the spin-orbit interaction; and $\mathbf{L}_i$ and $\mathbf{S}_i$ are the orbital and spin angular momentum operators for electron i.

The energy eigenvalues E_0 of H_0 are determined by the electron configuration. We will be interested in lead (Pb, $Z = 82$), with two $6p$ electrons outside closed subshells, and

bismuth (Bi, $Z = 83$), with three $6p$ electrons outside closed subshells. For such heavy atoms, H_2 is larger than H_1, so we first solve the Hamiltonian H_0, then successively add the terms H_2 and H_1, and see what happens to the energy levels and eigenstates.

We can assume that H_0 is solved for the ground state configuration, and that it gives a known energy E_0. This would involve Hartree-Fock theory, and would give us the central potential $\bar{V}_i$. Next we add H_2 and write

$$H_0 + H_2 = \sum_{i=1}^{Z} h(\mathbf{r}_i, \mathbf{p}_i, \mathbf{S}_i), \tag{1.2}$$

where

$$h(\mathbf{r}, \mathbf{p}, \mathbf{S}) = \frac{p^2}{2} - \frac{Z}{r} + \bar{V} + \xi(r)\mathbf{L} \cdot \mathbf{S}. \tag{1.3}$$

We write the single particle eigenstates and eigenvalues as

$$h|\lambda\rangle = \epsilon_\lambda |\lambda\rangle, \tag{1.4}$$

where λ stands for the single particle quantum numbers $(n\ell j m_j)$, and where ϵ_λ is independent of m_j, $\epsilon_\lambda = \epsilon_{nlj}$. The states $|\lambda\rangle$ form an orthonormal basis in the Hilbert space for a single electron. The eigenstates of $H_0 + H_2$ are Slater determinants formed out of Z orbitals $|\lambda\rangle$, and the corresponding energies are

$$E_{0+2} = \sum_\lambda \epsilon_\lambda. \tag{1.5}$$

We will only be interested in the ground state configuration of our atom (Pb or Bi), so the n and ℓ quantum numbers are the same for all orbitals λ, and the energy E_{0+2} depends only on the j quantum numbers.

For example, in Pb with the $6p^2$ configuration, the two electrons can give three possible pairs of j values, $(j_1, j_2) = (\frac{3}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$, or $(\frac{1}{2}, \frac{1}{2})$. Therefore the single level E_0 of H_0 breaks up into three levels E_{0+2}, labeled by (j_1, j_2), when we turn on H_2. As we know from spin-orbit theory (Notes 23), the energy $\epsilon_{n\ell j}$ is an increasing function of j, so the (j_1, j_2) are sequenced as indicated in Fig. 1.

Finally, when we turn on H_1, the individual $\mathbf{J}_i$ operators no longer commute with the Hamiltonian, but $\mathbf{J} = \sum_i \mathbf{J}_i$ does. Therefore we must now organize the energy eigenstates according to the (J, M_J) quantum numbers. In Pb, for example, the $(\frac{3}{2}, \frac{3}{2})$ state splits into $J = 0$ and $J = 2$ states. The states $J = 3$ and $J = 1$, which occur in $\frac{3}{2} \otimes \frac{3}{2}$, are not allowed by the Pauli principle. Likewise, the $(\frac{3}{2}, \frac{1}{2})$ level splits into $J = 1$ and $J = 2$ levels, because $\frac{3}{2} \otimes \frac{1}{2} = 1 \oplus 2$. Here the Pauli principle causes no extra restriction, because $j_1 \neq j_2$. Finally,

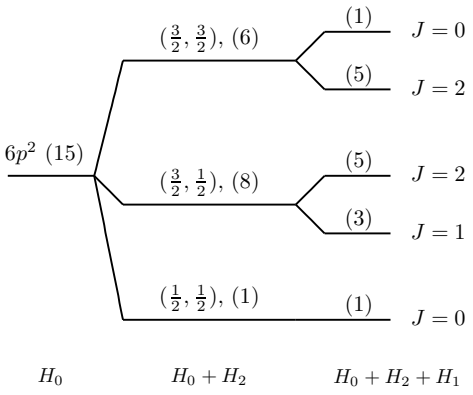

Fig. 1. Term diagram in jj-coupling for lead (Pb). Drawing is not to scale.

the $(\frac{1}{2}, \frac{1}{2})$ gives only a $J = 0$ level. All these levels and their degeneracies (in parentheses) are indicated in Fig. 1.

(a) For the case of bismuth ($6p^3$ configuration), indicate the allowed (j_1, j_2, j_3) terms and their degeneracies when H_2 is added to H_0. Make sure the degeneracies add up to the degeneracy of the $6p^3$ configuration. Indicate also the allowed J values contained in each (j_1, j_2, j_3) term. On the basis of the information given you can't know how to order the different J levels within a given (j_1, j_2, j_3) term, but the ordering shown in Fig. 1 are correct for Pb.

(b) The $(\frac{3}{2}, \frac{3}{2}, \frac{1}{2})$ term contains a $J = 3/2$ component. Find the normalized states $|JM_J\rangle$ in this component for $M_J = \frac{3}{2}$ and $M_J = \frac{1}{2}$. Write your answers as linear combinations of Slater determinants composed of orbitals $|\lambda\rangle$; the Slater determinant will be identified by the (j, m_j) quantum numbers of the last three orbitals, since the orbitals for the 80 core electrons are fixed. To help the grader(s), identify these Slater determinants as $|j_1 j_2 j_3; m_{j1} m_{j2} m_{j3}\rangle$.

2. (30 points) Emission of radiation by a neutron in a magnetic field.

(a) A neutron is initially at rest in a magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ in a spin-up state. Let M be the mass of the neutron. It emits a photon and drops into a spin-down state, and simultaneously recoils. Let ΔE be the energy difference due to the spin flip. Using simple conservation of energy and momentum, obtain a quadratic equation for the magnitude k

of the wave vector $\mathbf{k}$ of the photon in terms of ΔE. Use nonrelativistic mechanics for the neutron. It's a little messy to solve this quadratic for k so don't bother to do it, but do solve for k in the limit $M \to \infty$ (physically, this means $\Delta E \ll Mc^2$, a condition that would hold for reasonable magnetic fields). This part of the problem is easy.

(b) Sometimes in modeling we break the electromagnetic field into two parts, a classical or "external" or c-number field, and the quantized field. In the present case let $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, where $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ as in part (a) and where $\mathbf{B}_1$ is the quantized field,

$$\mathbf{B}_1(\mathbf{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_\lambda \frac{1}{\sqrt{\omega_k}} \big[i(\mathbf{k}\times\boldsymbol{\epsilon}_\lambda) a_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} - i(\mathbf{k}\times\boldsymbol{\epsilon}_\lambda^*) a_\lambda^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \big]. \tag{2.1}$$

This is essentially Eq. (39.20) of the Notes. Take the Hamiltonian for the entire system (neutron plus external field plus quantized field) to be

$$H = \frac{p^2}{2M} - \boldsymbol{\mu}\cdot(\mathbf{B}_0 + \mathbf{B}_1(\mathbf{x})) + \sum_\lambda \hbar\omega_\lambda\, a_\lambda^\dagger a_\lambda, \tag{2.2}$$

where $\mathbf{x}$ is the neutron position and $\boldsymbol{\mu}$ is its magnetic moment. For the convenience of the grader(s), please write the magnetic moment in the form

$$\boldsymbol{\mu} = -\mu_0 \frac{\mathbf{S}}{\hbar}, \tag{2.3}$$

where $\mu_0 > 0$ is a constant with dimensions of magnetic moment that can be expressed in terms of the neutron g-factor and the nuclear magneton.

Using first-order, time-dependent perturbation theory, find the differential transition rate $(dw/d\Omega)_\mu$ where $\mu = 1, 2$ or $-1, 1$ is the index of the polarization of the emitted photon (not to be confused with the magnetic moment). When you get to the point of taking into account conservation of energy, you may take $M \to \infty$ as in part (a).

(c) An observer on the x-axis views the emitted radiation through a linear polarizer. In a certain orientation of the polarizer the intensity of the radiation drops to zero. What is the axis of the polarizer when this happens? Does the emitted radiation as viewed from this direction have linear, elliptic, or circular polarization?

(d) Assuming we don't care about the polarization of the emitted radiation, find an expression for the differential transition rate $dw/d\Omega$. Integrate this to obtain the total transition rate w.

3. (40 points) We use natural units ($\hbar = c = 1$) in this problem.

Beta decay is the reaction

$$n \to p + e^- + \bar{\nu}, \tag{3.1}$$

the decay of a neutron into a proton, an electron, and an antineutrino. The Feynman diagram for this process is shown in Fig. 2.

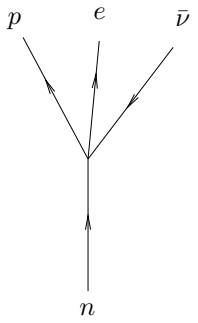

Fig. 2. Feynman diagram for β-decay.

In 1934 Fermi wrote down a field theory to explain β-decay. Since that time our understanding of weak interaction physics has become much more sophisticated (parity violation, helicity of neutrinos, weak currents, three types of neutrinos, electroweak unification, W and Z bosons, etc etc) but Fermi's theory does explain the basic experimental facts about β-decay (the energy spectrum of the emitted electron and other things) that were known at that time. In addition, Fermi's theory is accessible by methods developed in this course. Fermi's theory can be described by a field Hamiltonian,

$$H = H_{0n} + H_{0p} + H_{0e} + H_{0\nu} + H_{\rm int}, \tag{3.2}$$

where each term with a 0 subscript is the Hamiltonian for a free spin-$\frac{1}{2}$ fermion (neutron, proton, electron, neutrino), each with the general form

$$H_0 = \int d^3\mathbf{x} \,:\psi^\dagger(\mathbf{x})(-i\boldsymbol{\alpha}\cdot\nabla + m\beta)\psi(\mathbf{x}) := \sum_{ps} E(b^\dagger_{ps}b_{ps} + d^\dagger_{ps}d_{ps}), \tag{3.3}$$

where H_0, ψ, m, E, b_{ps}, d_{ps} etc all implicitly take a subscript $i = n, e, p, \nu$ to indicate the type of particle. The notation is as in class and in the notes. The fermion field ψ has the Fourier expansion,

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \Big(b_{ps} u_{ps} e^{i\mathbf{p}\cdot\mathbf{x}} + d^\dagger_{ps} v_{ps} e^{-i\mathbf{p}\cdot\mathbf{x}} \Big), \tag{3.4}$$

where V is the volume of the box and otherwise the notation is as in class.

As for the interaction Hamiltonian in Eq. (3.2), Fermi chose it to be the product of four Fermion fields in order to take into account the 4-point vertex in Fig. 2. He also chose a current-current type of interaction, since the electromagnetic interaction is of this type. Fermi's interaction Hamiltonian is

$$H_{\rm int} = g \int d^3\mathbf{x} \, : (\bar\psi_p \gamma^\mu \psi_n)(\bar\psi_e \gamma_\mu \psi_\nu) : + \text{h.c.}, \tag{3.5}$$

where g is a constant, where all fields are evaluated at $\mathbf{x}$ and where h.c. means "Hermitian conjugate." By using this Hamiltonian to calculate the rate of neutron decay and comparing to experiment one can get a value for the constant g (essentially, the Fermi constant).

(a) Explain why the Hermitian conjugate of a term such as $\bar\psi_p \gamma^\mu \psi_n$ is $\bar\psi_n \gamma^\mu \psi_p$ (it is not completely obvious since we have $\bar\psi$ instead of $\psi^\dagger$). This means that the Hermitian conjugate term in the Hamiltonian can be written

$$g \int d^3\mathbf{x} \, : (\bar\psi_\nu \gamma_\mu \psi_e)(\bar\psi_n \gamma^\mu \psi_p) : . \tag{3.6}$$

(b) Show that the interaction Hamiltonian connects the initial and final states in Fig. 2, that is, the matrix element $\langle f | H_{\rm int} | i \rangle$ is nonzero. For this it is sufficient to show that the field part of the matrix element is nonzero. You don't have to write out the matrix element in all detail, you can be schematic, but indicate the important parts for the question at hand. Notice that the interaction gives rise to $2 \times 16 = 32$ possible Feynman diagrams. Show that the Feynman diagram for inverse β-decay,

$$p + e \to n + \nu \tag{3.7}$$

is one of them. Draw a Feynman diagram connecting the four particles and their antiparticles that is *not* one of the 32.

(c) Inverse β-decay (3.7) takes place in the final stages of core collapse in a supernova explosion. As the matter is compressed by gravity the top of the electron Fermi sea rises, ultimately reaching relativistic energies ($\sim$ 511 KeV) and beyond. When it reaches $\sim$ 1.3 MeV, the mass difference between the neutron and the proton, the electrons are energetic enough to cause the reaction (3.7) to take place. Any extra electron energy goes mostly into the energy of the neutrino. As the most energetic electrons are removed, their contribution to the pressure is also eliminated, leading to further gravitational contraction. Simultaneously, protons are converted to neutrons and neutrinos are emitted. In this way a good part

of the neutrons in a neutron star are created. The neutrinos from a supernova explosion were actually observed in 1989.

We will use Fermi's theory to compute the cross section for inverse beta decay. Before beginning the quantum mechanics, it's a good idea to practice a little with the conservation laws. Working in the center of mass frame, write down an expression for the total energy $E_{\rm tot}$ of the system as a function of the initial electron 3-momentum $p_e = |\mathbf{p}_e|$. The momentum $\mathbf{p}_e$ is a parameter of the problem and is fixed for the rest of the calculation; therefore so is $E_{\rm tot}$. Now compute $E_{\rm tot}$ as a function of the final neutrino 3-momentum $p_\nu = |\mathbf{p}_\nu|$. This final 3-momentum $\mathbf{p}_\nu$ should be regarded as a variable, since we will be summing over a collection of final states to get a cross section. Imagine solving for p_ν as a function of $E_{\rm tot}$. Don't actually do it, since it's a little messy, but call the root obtained p_f (for "final" momentum). In the limit $m_\nu \to 0$, however, it's easy to solve for p_ν as a function of $E_{\rm tot}$; do this. The whole calculation is done in the center-of-mass frame.

(d) Find an expression for the matrix element $\langle f|H_{\rm int}|i\rangle$ that is simplified as much as you can make it without explicitly evaluating spinors or spin contractions. Work with box normalization as in class and as reflected by the equations above.

(e) Now write out an expression for the differential cross section $d\sigma/d\Omega$, where $d\Omega$ refers to a cone of small solid angle in some direction, within which the final neutrino momentum lies. Simplify this as much as you can without doing spin sums. You may express things in terms of the momentum p_f found in part (c) and other convenient quantities. Hint: the incident flux can be defined as the number of electrons per unit volume in the initial state times the relative velocities of the initial electron and proton.

(f) Now assume that the incident electron and proton are unpolarized, and that we do not care about the spins of the outgoing neutron and neutrino. Write an expression for the effective differential cross section in this case, and take the limit $m_\nu \to 0$. At the same time you may assume that the proton and neutron are essentially nonrelativistic, so $E \approx m$ and $v \ll 1$ for these particles. This is appropriate for the astrophysical application discussed above. Do the spin sums to get a practical formula for the effective $d\sigma/d\Omega$.

(g) Integrate this to get a total cross-section.