Now on to magnetic monopoles. Let a M.M. be located at \( \mathbf{r} = 0 \). Then

\[ \mathbf{B} = \mu \frac{\mathbf{r}}{r^2} \]

\( \mu \) = strength of monopole.

and

\[ \nabla \cdot \mathbf{B} = 4\pi \mu \delta(\mathbf{r}) \].

So Maxwell eqn. \( \nabla \cdot \mathbf{B} = 0 \) no longer valid, except in region \( r > 0 \) (we exclude small region around monopole itself.). Since this region is not contractible, we suspect difficulty in constructing \( \mathbf{A} \).

If you unravel \( \mathbf{A} \) in spherical coordinates, one obvious solution (there are others) is

\[ \mathbf{A} = \mu \frac{(1 - \cos \theta) \hat{\phi}}{r \sin \theta} \]

This is singular on the neg. \( z \)-axis \( (\theta = \pi) \) but not on pos. \( z \)-axis, \( \sin \theta \approx \theta \), \( (\theta = 0) \) because \( 1 - \cos \theta \approx \theta ^2 / 2 \) when \( \theta \ll 1 \). The singularity on the neg. \( z \)-axis is called the string of the monopole.

It is really the string of the vector potential.

One can show that there does not exist any smooth, singularity-free \( \mathbf{A} \) in the region \( r > 0 \), such that \( \mathbf{B} = \nabla \times \mathbf{A} = \mu \hat{r} / r^2 \).
Suppose such an $\mathbf{A}$ exists. Consider sphere surrounding monopole with small hole removed. $S =$ surface of sphere, $\partial S =$ edge of hole.

By Stokes' Theorem,
\[ \int_{\partial S} \mathbf{B} \cdot d\mathbf{a} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}. \]

Now take limit as hole $\to 0$:

\[ \int_{\partial S} \mathbf{B} \cdot d\mathbf{a} = 4\pi \mu = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = 0 \quad \text{if} \quad \mathbf{A} \quad \text{smooth.} \]

Contradiction, so $\mathbf{A}$ cannot be smooth everywhere. The singularity (string) we have in the choice of gauge above cannot be eliminated by any smooth gauge transformation.

But it can be moved around. Let $f = -2\mu \phi$, where $\phi =$ usual azimuthal angle. Then
\[ \nabla f = -2\mu \frac{1}{r \sin \theta} \hat{\phi}, \]

so,
\[ \mathbf{A}' = \mathbf{A} + \nabla f = -\mu \frac{(1 + \cos \theta)}{r \sin \theta} \hat{\phi}. \]
This vector potential is singular on pos z-axis (the string), well-behaved on neg:

\[ \vec{A}_N = \mu \frac{(1 - \cos \theta)}{r \sin \theta} \hat{\phi} \quad \text{smooth in Northern hemisphere singular on S pole.} \]

\[ \vec{A}_S = -\mu \frac{(1 + \cos \theta)}{r \sin \theta} \hat{\phi} \quad \text{smooth in Southern Hemisphere singular at N pole.} \]

Call these two gauges N and S:

There is a strip around the equator where both gauges are valid, and

\[ \vec{A}_N = \vec{A}_S + \nabla (2\mu \phi). \]
Monopoles, cont'd. We are interested in the QM of a charged particle in the presence of a monopole field.

**Summary:** \( \mathbf{B} = \mu \frac{\hat{r}}{r^2}, \quad \nabla \cdot \mathbf{B} = 4\pi \mu \varepsilon(\mathbf{r}), \quad \mu = \text{strength of monopole.} \)

In region \( r > 0 \), \( \nabla \cdot \mathbf{B} = 0 \):

\[
\begin{align*}
\mathbf{A}_N &= \mu \frac{(1 - \cos \theta)}{r \sin \theta} \hat{\phi} \\
\mathbf{A}_S &= -\mu \frac{(1 + \cos \theta)}{r \sin \theta} \hat{\phi}
\end{align*}
\]

\( \nabla \times \mathbf{A}_N = \nabla \times \mathbf{A}_S = \mathbf{B} \)

\( \mathbf{A}_N = \mathbf{A}_S + \nabla \phi \) where \( \phi = 2\mu \phi \) (usual \( \phi \) = azimuth angle).

\( \mathbf{A}_N \) valid in N hemisphere, pushing into S

\( \mathbf{A}_S \) valid in S hemisphere, pulling into N.

\( \mathbf{A}_N, \mathbf{A}_S \) have a common region of validity, a strip around the equator. There are also 2 wavefuns \( \psi_N, \psi_S \), defined over the two regions. In the overlap region, they are related by the gauge transformation,

\[
\psi_N = e^{\frac{i}{\hbar} \frac{e}{2} (2\mu \phi)} \psi_S
\]

(charge of particle = \( e \))
The wave fn. must be single valued. This applies to both $\Psi_1$ and $\Psi_2$ in their respective domains of definition, including the overlap region. Thus the phase factor

$$e^{\frac{2\imath e\Phi}{\hbar c} \phi}$$

must be periodic under $\phi \rightarrow \phi + 2\pi$, i.e.,

$$\frac{2\imath e\Phi}{\hbar c} = n = \text{integer}.$$ 

The product $e\Phi$ must be quantized in order for QM to be consistent. If we solve for $e\Phi$, this means

$$\mu = \frac{n m}{2} \frac{\hbar e}{c},$$

or

$$\Phi = \text{flux of monopole} = \oint \mathbf{B} \cdot d\mathbf{A} = 4\pi \mu = n \frac{\hbar e}{c} = n \Phi_0,$$

where $\Phi_0$ is the flux quantum. Or, solving for $e$,

$$e = \frac{n}{2} \frac{\hbar c}{\mu}.$$ 

Dirac's argument: If a monopole exists anywhere in the universe, then electric charge is quantized. In fact, electric charge is quantized. Do this the explanation? No one knows (monopoles have not been discovered.)