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Now on to <u>magnetic monopoles</u>. Let a M.M. be located at $\vec{r}=0$. Then

$$\vec{B} = \mu \frac{\hat{r}}{r^2}$$

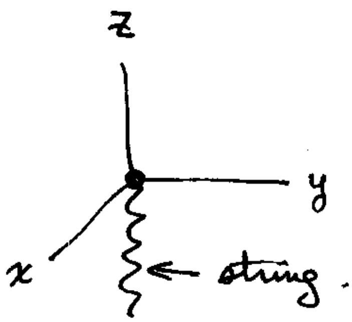
μ = strength of monopole.

and $\nabla\cdot\vec{B} = 4\pi\mu\,\delta(\vec{r})$. So Maxwell eqn. $\nabla\cdot\vec{B}=0$ no longer valid, except in region $r>0$ (we exclude small region around monopole itself.) Since this region is not contractible, we suspect difficulty in constructing $\vec{A}$.

If you uncurl $\vec{A}$ in spherical coordinates, one obvious solution (there are others) is

$$\vec{A} = \mu \frac{(1-\cos\theta)}{r\sin\theta}\,\hat{\phi}.$$

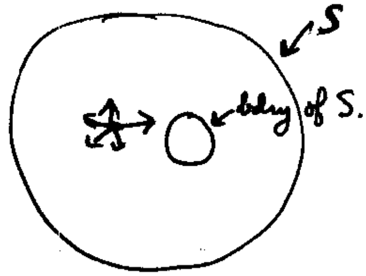
This is singular on the neg. z-axis $(\theta=\pi)$ but not on pos z-axis, $(\theta=0)$ because $1-\cos\theta \approx \theta^2/2$, $\sin\theta \approx \theta$, when $\theta \ll 1$. The singularity on the neg. z-axis is called the <u>string</u> of the monopole.



It is really the string of the vector potential.

One can show that there does not exist any smooth, singularity-free $\vec{A}$ in the region $r>0$, such that $\vec{B} = \nabla\times\vec{A} = \mu\,\hat{r}/r^2$.

Suppose such an $\vec{A}$ exists. Consider sphere surrounding monopole with small hole removed. S = surface of sphere, bdry of S = edge of hole.



By Stokes' thm,

$$\int_{\text{surf. } S} \vec{B}\cdot d\vec{a} = \oint_{\text{Bdry}} \vec{A}\cdot d\vec{l}.$$

Now take limit as hole $\to 0$:

$$\int_{\text{sphere}} \vec{B}\cdot d\vec{a} = 4\pi\mu = \oint_{\text{Bdry}\to 0} \vec{A}\cdot d\vec{l} = 0 \quad \text{if } \vec{A} \text{ smooth.}$$

Contradiction, so $\vec{A}$ cannot be smooth everywhere. The singularity (string) we have in the choice of gauge above ~~[illegible]~~ cannot be eliminated by any smooth gauge transformation.

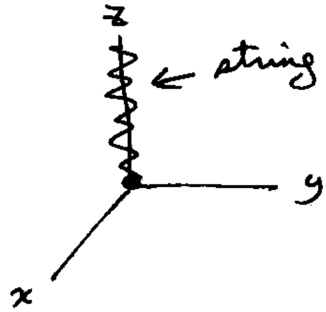
But it the string can be moved around. Let $f = -2\mu\phi$, where ϕ = usual azimuthal angle. Then

$$\nabla f = -2\mu \frac{1}{r\sin\theta}\hat{\phi},$$

So,

$$\vec{A}' = \vec{A} + \nabla f = -\mu\frac{(1+\cos\theta)}{r\sin\theta}\hat{\phi}.$$

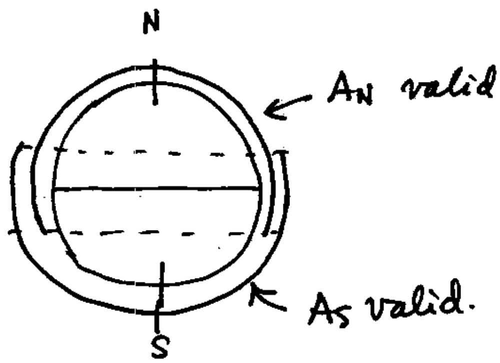
This vector potential is singular on pos z-axis (the string), well-behaved on neg:



Call these two gauges N and S:

$$\vec{A}_N = \mu \frac{(1-\cos\theta)}{r\sin\theta}\hat{\phi}$$ smooth in Northern hemisphere singular on S pole.

$$\vec{A}_S = -\mu \frac{(1+\cos\theta)}{r\sin\theta}\hat{\phi}$$ Smooth in Southern Hemisphere singular at N pole.



There is a strip around the equator where both gauges are valid, and

$$\vec{A}_N = \vec{A}_S + \nabla(2\mu\phi).$$

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Monopoles, cont'd. We are interested in the QM of a charged particle in the presence of a monopole field.

Summary: $\vec{B} = \mu \frac{\hat{r}}{r^2}$, $\nabla \cdot \vec{B} = 4\pi\mu\,\delta(\vec{r})$, μ = strength of monopole.

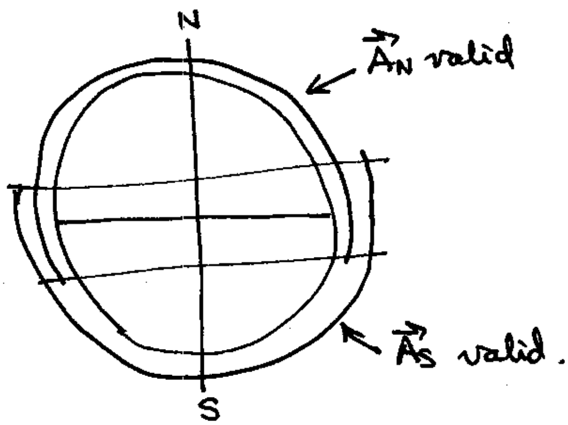
In region $r > 0$, $\nabla \cdot \vec{B} = 0$

$$\left\{ \begin{aligned} \vec{A}_N &= \mu \frac{(1-\cos\theta)}{r\sin\theta}\hat{\phi} \\ \vec{A}_S &= -\mu \frac{(1+\cos\theta)}{r\sin\theta}\hat{\phi} \end{aligned} \right\}. \qquad \nabla \times \vec{A}_N = \nabla \times \vec{A}_S = \vec{B}$$

$$\vec{A}_N = \vec{A}_S + \nabla f \quad \text{where } f = 2\mu\phi \qquad (\phi = \text{usual azim. angle}).$$

$\vec{A}_N$ valid in N hemisphere, pushing into S

$\vec{A}_S$ " " S " , " " N.



$\vec{A}_N$, $\vec{A}_S$ have a common region of validity, a strip around the equator. There are also 2 wavefns ψ_N, ψ_S, defined over the two regions. In the overlap region, they are related by the gauge transformation,

$$\psi_N = e^{\frac{i}{\hbar}\frac{e}{c}(2\mu\phi)}\,\psi_S \qquad \text{(charge of particle} = e)$$

The wave fn. must be singled valued. This applies to both ψ_N and ψ_S in their respective domains of definition, including the overlap region. Thus the phase factor

$$e^{\frac{2ie\mu}{\hbar c}\phi}$$

must be periodic under $\phi \to \phi + 2\pi$, i.e.,

$$\frac{2e\mu}{\hbar c} = n = \text{integer}.$$

The product $e\mu$ must be quantized in order for QM to be consistent. If we solve for μ, this means

$$\mu = \frac{n}{2}\frac{\hbar c}{e},$$

or

$$\Phi = \text{flux of monopole} = \oint_{\text{sphere}} \vec{B}\cdot d\vec{a} = 4\pi\mu = n\,\frac{hc}{e} = n\,\Phi_0,$$

where Φ_0 is the flux quantum. Or, solving for e,

$$e = \frac{n}{2}\frac{\hbar c}{\mu}.$$

<u>Dirac's argument</u>: If a monopole exists anywhere in the universe, then electric charge is quantized. In fact, electric charge <u>is</u> quantized. Is this the explanation? No one knows (monopoles have not been discovered.)