

(1)

We have found the spectrum (eigenvalues of  $H$ ) for the particle in the uniform  $\vec{B}$  field without ever choosing a gauge convention for  $\vec{A}$ , that is, we did it in a gauge-invariant manner. It is logical we can do this, since the energy eigenvalues are physically measurable and must be gauge-invariant.

We now turn to the energy eigenfunctions, the solutions of

$$H \Psi(x, y, z) = E \Psi(x, y, z)$$

which will be gauge-dependent and so we will have to choose a gauge. We must do something else, too, which is to choose a 3rd operator, which along with  $H_1$  and  $\hat{P}_3$ , will form a complete set. This is because the simultaneous eigenstates of  $H_1$  and  $\hat{P}_3$  are degenerate. It is logical that we should need a 3rd operator, since  $H_1$  depends only on  $Q_1, P_1$  and  $H_2$  depends only on  $P_3$ . The #1 degree of freedom  $(Q_1, P_1)$  or  $(X, Y)$  does not appear.

It's obvious any function of  $X, Y$  could be used for the 3rd operator; in these notes we'll take it to be  $X$ . In a HW problem, you will look at the case that the 3rd operator is the angular momentum. That is we will take

$$\text{C.S.C.O.} = (\hat{X}, H_1, \hat{P}_3) \quad \begin{matrix} \text{(putting hats on} \\ \text{some operators)} \end{matrix}$$

We'll show this is a CSCO by showing that the simultaneous

(2)

eigenfunctions are nondegenerate.

10/2/07

We really have to solve 3 Schrödinger equations,

$$\hat{X} \psi(x, y, z) = X \psi(x, y, z)$$

$$\hat{H}_\perp \psi(x, y, z) = E_\perp \psi(x, y, z)$$

$$\hat{P}_3 \psi(x, y, z) = P_3 \psi(x, y, z)$$

The best strategy in cases like this is to diagonalize the easy operators first, and to leave the hardest operator (usually the Hamiltonian) for last. We'll start with  $\hat{P}_3 = m\hat{\partial}_z$ . We must solve

operator  $\hat{P}_3 \psi(x, y, z) = P_3 \psi(x, y, z)$   $\downarrow$  eigenvalue

$$\hat{P}_3 \psi(x, y, z) = P_3 \psi(x, y, z)$$

$$= m\hat{\partial}_z \psi = (P_z + \frac{e}{c}A_z)\psi$$

$$= \left( -i\hbar \frac{\partial}{\partial z} + \frac{e}{c} A_z(x, y, z) \right) \psi(x, y, z).$$

The presence of  $A_z(x, y, z)$  makes this hard. Can we choose a gauge such that  $A_z = 0$ ? Look at  $\vec{B} = \nabla \times \vec{A} = B_z \hat{z}$  in components:

$$B = B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

$$0 = B_x = \cancel{\frac{\partial A_z}{\partial y}} - \frac{\partial A_y}{\partial z}$$

$$0 = B_y = \frac{\partial A_x}{\partial z} - \cancel{\frac{\partial A_z}{\partial x}}$$

If  ~~$B_z$~~   $A_z = 0$ , then 2 terms cancel (shown), and to make  $B_x = B_y = 0$

(3)

10/2/07

we must have  $\frac{\partial A_x}{\partial z} = \frac{\partial A_y}{\partial z} = 0$ . so  $\vec{A}$  becomes

$$\vec{A} = A_x(x,y) \hat{x} + A_y(x,y) \hat{y}$$

which still leaves plenty of freedom to make  $B_z = \nabla \times \vec{A}$ . so let's take  $A_z = 0$  (a gauge convention). Then  $\hat{p}_z = \text{canon. mom.} = m\hat{v}_z = \hat{P}_3$ , so  $\hat{P}_3$  is both the kin. and can. momentum in the  $z$ -direction. So let's change notation,

$$\hat{P}_3 \rightarrow \hat{p}_z$$

$$P_3 \rightarrow p_z$$

and the eigenvalue eqn. is

$$-i\hbar \frac{\partial}{\partial z} \psi(x,y,z) \xrightarrow{\text{eigenvalue}} \hat{p}_z \psi(x,y,z),$$

or

$$\psi(x,y,z) = \phi(x,y) e^{ip_z z/\hbar},$$

where  $\phi(x,y)$  is an arbitrary fn of  $(x,y)$ . This is the general sol'n of the  $\hat{p}_z$ -eqn. We must determine  $\phi(x,y)$  by demanding that  $\psi(x,y,z)$  also be an eigenfn. of  $\hat{H}_1$  and  $\hat{x}$ .

We take  $\hat{x}$  next. We require

operator  $\hat{x} \psi(x,y,z) = \xleftarrow{\text{eigenvalue}} \hat{x} \psi(x,y,z)$

$$= (\hat{x} - \hat{v}_y/\omega) \psi = \left[ \hat{x} - \frac{1}{m\omega} \left( \hat{p}_y + \frac{e}{c} A_y \right) \right] \psi$$

(4)  
10/2/07

choose  $A_y$  to cancel

$$= \left[ x - \frac{1}{m\omega} \left( -i\hbar \frac{\partial}{\partial y} + \frac{e}{c} A_y(x, y) \right) \right] \psi(x, y, z) = X \psi(x, y, z).$$

We'll simplify this by choosing  $A_y$  to cancel all the  $x$ - $y$  dependence on the left, i.e.

$$x = \frac{e}{m\omega c} A_y \quad \text{or} \quad A_y = \frac{mc}{e} \omega x \\ = \frac{mc}{e} \frac{eB}{mc} x$$

or  $A_y = Bx$ . Plugging this into  $B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$

we see it will work if  $A_x = 0$ . Altogether, we have the choice of gauge,

$$\vec{A} = Bx \hat{y} \quad (A_x = A_z = 0).$$

Then the  $x$  eigenvalue problem becomes

$$\frac{i\hbar}{m\omega} \frac{\partial}{\partial y} (\phi(x, y) e^{ip_z z/\hbar}) = X (\phi(x, y) e^{ip_z z/\hbar})$$

or cancelling the  $z$ -dependence,

$$\frac{i\hbar}{m\omega} \frac{\partial \phi}{\partial y} = X \phi,$$

or

$$\phi(x, y) = f(x) e^{-\frac{im\omega X}{\hbar} y},$$

(5)

10/2/07

where  $f$  is an arbitrary fn. of  $x$ . Now we have

$$\psi(x, y, z) = f(x) e^{i(k_y y + k_z z)},$$

where

$$\left. \begin{aligned} k_y &= -\frac{m\omega X}{\hbar} \\ k_z &= p_z/\hbar \end{aligned} \right\}.$$

The soln is a plane wave in the  $y$  and  $z$  directions.

Finally we make  $\psi$  an eigenfn. of  $H_{\perp}$ :

$$H_{\perp} \psi(x, y, z) = E_{\perp} \psi(x, y, z)$$

$$= \frac{m}{2} (\hat{v}_x^2 + \hat{v}_y^2) \psi = \frac{1}{2m} \left[ (p_x + \frac{e}{c} A_x)^2 + (p_y + \frac{e}{c} A_y)^2 \right] \psi.$$

But  $A_x = 0$  and  $A_y = Bx$ , so  $\frac{e}{c} A_y = m\omega X$ . Also, when  $p_y = -i\hbar \frac{\partial}{\partial y}$  acts on  $\psi(x, y, z) = f(x) e^{i(k_y y + k_z z)}$ , it brings down a factor of  $i\hbar k_y = -m\omega X$ . So we get

$$\left[ \frac{p_x^2}{2m} + \frac{m\omega^2}{2} (x - X)^2 \right] f(x) = E_{\perp} f(x),$$

after cancelling the factor  $e^{i(k_y y + k_z z)}$ . This is a 1D H.O. eqn. with shifted origin (origin at  $x=X$  instead of  $x=0$ ), so the solutions are

$$f(x) = u_n(x-X),$$

(6)

10/2/07

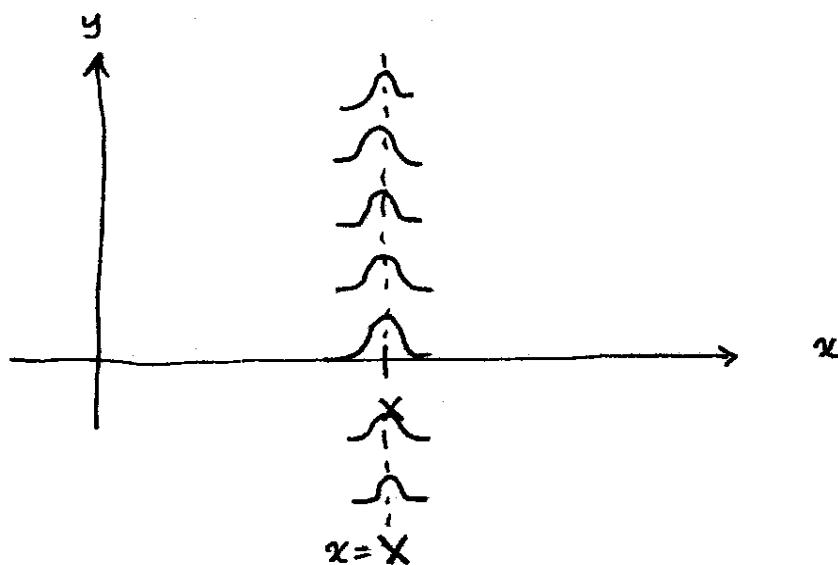
where  $u_n(x)$  is the normalized H.O. eigenfunction for a 1D oscillator of mass  $m$  and freq.  $\omega$ . The eigenvalues are  $E_L = (n + \frac{1}{2})\hbar\omega$  (the Landau levels).

The overall wavefn. is parameterized by 3 quantum numbers,  $(X, n, p_z)$ , and it is

$$\psi_{Xnp_z}(x, y, z) = u_n(x-X) e^{-i \frac{m\omega X}{\hbar} y + i p_z z / \hbar}$$

in the gauge  $\vec{A} = Bx \hat{y}$ .

If we plot  $|\psi|^2 = u_n(x-X)^2$  in the  $x-y$  plane, it is a "mountain ridge" along the line  $x=X$ . For example, in the  $n=0$  (ground Landau) state, the mountain range looks like this:



Effectively, the classical orbit for given  $X$  is smeared out in  $y$ , for all  $-\infty < y < +\infty$ . This is because  $[x, y] = \frac{i\hbar}{m\omega} \neq 0$ , so an eigenstate of  $X$  is completely undetermined in  $y$ .