Now charged particle motion in a uniform $\mathbf{B}$ field, a problem closely related to harmonic oscillators and important in applications. First, two remarks however.

**First remark.** The HO Hamiltonian for a mechanical HO is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$

This can be put into a more symmetrical form by the transformation,

$$p = \sqrt{m\omega} \, p',
\quad x = \frac{1}{\sqrt{m\omega}} \, x',$$

which preserves the canonical commutation relations,

$$[x, p] = [x', p'] = i\hbar.$$

In terms of the new variables,

$$H = \frac{\omega}{2} (x'^2 + p'^2).$$

The second remark concerns velocity operators. The Hamiltonian for a charged particle in an EM field is

$$H = \frac{1}{2m} \left[ \mathbf{p} - \frac{q}{m} \mathbf{A} \right]^2 + q \phi. \quad q = \text{charge of particle.}$$

The first term is physically just the kinetic energy. We define the velocity operator (a vector of operators) by

$$\mathbf{v} = \frac{1}{m} \left( \mathbf{p} - \frac{q}{m} \mathbf{A} \right).$$
This definition is borrowed from classical mechanics, or else we can make a purely quantum motivation by invoking the Heisenberg equations of motion,

\[ \dot{x} = \frac{\dot{p}}{m} = \frac{i}{\hbar} [x, H] = \frac{i}{\hbar} (\vec{p} - \frac{e\vec{A}}{c}) . \]

The velocity operators do not commute, in general. Using

\[ [x_i, p_j] = i\hbar \delta_{ij} , \]

we find

\[ [v_i, v_j] = \frac{1}{m^2} \left[ p_i - \frac{e}{c} A_i, p_j - \frac{e}{c} A_j \right] \]

\[ = -\frac{e}{m^2 c} \left( [p_i, A_j] + [A_i, p_j] \right) \]

\[ = +\frac{i\hbar q}{m^2 c} \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \]

or,

\[ [v_i, v_j] = \frac{i\hbar q}{m^2 c} \varepsilon_{ijk} B_k \]

The velocity components commute if \( \vec{B} = 0 \), otherwise not.

Also, we note the commutators,

\[ [x_i, v_j] = \frac{i\hbar}{m} \delta_{ij} \]

In terms of the velocity operators, the Hamiltonian is
\[ H = \frac{1}{2} m v^2 + q \phi, \]

which is more transparent than the usual expression (in terms of the canonical momentum \( \mathbf{p} \)).

Now let's set \( \mathbf{E} = 0 \), \( \mathbf{B} = \mathbf{B} \hat{z} \) (\( B = \text{const} \)), and \( q = -e \) (where \( e > 0 \), that is, a negative particle). First we solve the classical problem. The Newton-Lorentz equations are

\[ m \ddot{\mathbf{a}} = -e \frac{\mathbf{v} \times \mathbf{B}}{c} \]

or

\[ \ddot{\mathbf{a}} = \frac{eB}{mc} \mathbf{v} \times \mathbf{B} \]

or

\[ \ddot{\mathbf{a}} = \omega \hat{\mathbf{z}} \times \mathbf{v} \]

where

\[ \omega = \frac{eB}{mc} > 0 \quad (\text{gyrofrequency}). \]

Integrating once, we get

\[ \mathbf{v} = \omega \hat{\mathbf{z}} \times \mathbf{r} + \mathbf{c}, \]

where \( \mathbf{c} \) is a constant vector. Writing out all 3 components, we have

\[
\begin{align*}
\dot{v}_x &= -\omega y + c_x \\
\dot{v}_y &= +\omega x + c_y \\
\dot{v}_z &= c_z
\end{align*}
\]

The particle velocity is constant in the \( z \)-direction, that is,
it is free-particle motion. We'll just write \( \nu_x \) for the \( z \)-velocity (not \( c_z \)), and remember that it is constant. As for the \( x \) and \( y \) components, let's set

\[
\begin{align*}
\nu_x &= \omega Y \\
\nu_y &= -\omega X,
\end{align*}
\]

where \( X \) and \( Y \) are constants that we use instead of \( c_x, c_y \), so we have

\[
\begin{align*}
\nu_x &= -\omega (y - Y) \\
\nu_y &= \omega (x - X),
\end{align*}
\]

or

\[
\begin{align*}
X &= x - \nu_y / \omega \\
Y &= y + \nu_x / \omega.
\end{align*}
\]

We interpret \( X, Y \) physically in a moment.

Now define

\[
\begin{align*}
\xi &= x - X & \xi &= \dot{x} = \nu_x \\
\eta &= y - Y & \eta &= \dot{y} = \nu_y
\end{align*}
\]

then

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \omega
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}.
\]

This can be solved by exponentiating the matrix,
\[ \exp \left[ \omega t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}, \]

which is a rotation matrix in the x-y plane in a positive (counterclockwise) direction. Thus the solution is

\[ \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}, \]

or in pictures,

Or, since

\[ x = X + \xi \]
\[ y = Y + \eta \]

the motion in the x-y plane is \( (x(t), y(t)) \)
Constants $X$ and $Y$ are the center of the circle, which is called the guiding center (or its projection onto the $xy$ plane, if we include the $z$-motion). The motion in 3d space is a helix.

Now we turn to the quantum problem. First we find the spectrum of $H$, which is easier than finding the wave functions. The energy eigenvalues are gauge-invariant (they do not depend on the choice of vector potential), and we shall determine them by gauge-invariant means.

The Hamiltonian is (in terms of velocity operators)

$$H = \frac{1}{2} m \mathbf{v}^2 = \frac{1}{2} m (v_x^2 + v_y^2) + \frac{1}{2} m v_z^2 = H_\perp + H_{11}.$$ 

The energy is purely kinetic, and is broken into perpendicular (to \( \overline{B} \)) and parallel (to \( \overline{B} \)) parts. Since \( \overline{B} = B^z \) and \( q = -e \), the velocity commutators are

$$[v_x, v_y] = \frac{-i \hbar e B}{m^2 c} = \frac{-i \hbar \omega}{m},$$

$$[v_x, v_z] = [v_y, v_z] = 0.$$ 

We see immediately that $[H_\perp, H_{11}] = 0$, so they possess simultaneous eigenstates.

Our strategy for analyzing $H$ is to borrow definitions...
of physically interesting operators from the classical solution and to explore their properties. The most interesting operators for the I motion are $X, Y$ and $\hat{v}_x, \hat{v}_y$, while for the II motion they are $\hat{z}, \hat{v}_z$. The definitions of $X, Y$ (the guiding center operators) are

$$X = x - \frac{v_y}{\omega}$$
$$Y = y + \frac{v_x}{\omega}$$

exactly as in the classical case, but now interpreted as operator equations. Later we will have to distinguish between operators and c-numbers (classical values or eigenvalues), so we will write $\hat{X}, \hat{Y}$ etc. vs. $X, Y$ etc. when we want to make the distinction clear.

The commutator of $\hat{X}$ and $Y$ is

$$[X, Y] = [x - \frac{v_y}{\omega}, y + \frac{v_x}{\omega}] = \frac{i\hbar}{m\omega}$$

$$= \frac{1}{\omega} [x, \hat{v}_x] - \frac{1}{\omega} [\hat{v}_y, y] + \frac{1}{\omega^2} [\hat{v}_y, \hat{v}_x]$$

Also, by similar calculations we find that $X$ and $Y$ commute with $\hat{v}_x, \hat{v}_y$. Including the $\hat{z}, \hat{v}_z$ variables, we get an overall set of commutation relations,
\[ [X, Y] = \frac{i \hbar}{m \omega} \]

\[ [X, v_x] = [X, v_y] = [X, z] = [X, v_z] = 0 \]

\[ [Y, v_x] = [Y, v_y] = [Y, z] = [Y, v_z] = 0 \]

\[ [v_y, v_x] = \frac{i \hbar \omega}{m} \]

\[ \uparrow \]

\[ \text{swapped order} \]

\[ [v_x, z] = [v_x, v_z] = [v_y, z] = [v_y, v_z] = 0 \]

\[ [z, v_z] = \frac{i \hbar}{m} . \]

We have 6 operators, \( X, Y, v_x, v_y, z, v_z \) with 3 nonvanishing commutators proportional to \( i \hbar \), and all others zero. To within constant factors, it is a set of 3 pairs of conjugate variables.

Let us therefore define:

\[
\begin{align*}
X &= \frac{1}{\sqrt{m \omega}} \, Q_1 \\
Y &= \frac{1}{\sqrt{m \omega}} \, P_1 \\
v_y &= \sqrt{\frac{\omega}{m}} \, Q_2 \\
v_x &= \sqrt{\frac{\omega}{m}} \, P_2 \\
z &= Q_3 \\
m v_z &= P_3
\end{align*}
\]

Then:

\[ [Q_i, Q_j] = [P_i, P_j] = 0 \]

\[ [Q_i, P_j] = i \hbar \delta_{ij} . \]
By working with variables of obvious physical importance, we have found a "canonical transformation", 
\((x, y, z, p_x, p_y, p_z) \rightarrow (Q, Q_x, Q_y, P, P_x, P_y, P_z)\), i.e. one which preserves the canonical commutation relations.

Now we express \(H\) in terms of these new variables.

We have

\[ H = \frac{\omega}{2} \left( Q_x^2 + P_x^2 \right) + \frac{P_z^2}{2m} \]

The operator \(H_{\perp}\) is a harmonic oscillator, with eigenvalues \((n + \frac{1}{2})\hbar \omega\). These are called Landau levels (quantized states of \(\pm\) kinetic energy). Operator \(H_{||}\) is a 1d free particle Hamiltonian. The energies are

\[ E = E_{\perp} + E_{||} \]

\[ = (n + \frac{1}{2})\hbar \omega + \frac{P_z^2}{2m} \]

This Hamiltonian is unusual because it is independent of both \(Q_x, Q_y\) and \(P_x, P_y\) (essentially the guiding center positions \(X, Y\)). This makes sense, since the classical energy is independent of \(X, Y\) (the circle can be centered anywhere in the \(x-y\)-plane without changing the energy).