Physics 221A
Fall 2007
Homework 7
Due Friday, October 19, 2007

Reading Assignment: Sakurai, pp. 123–143, 152–155; rest of Notes 8, Notes 9 through p. 10; handwritten lecture notes.

1. The path integral for the harmonic oscillator.

(a) Sakurai (p. 116) says that there is a unique classical path connecting two endpoints and endtimes, (x_0, t_0), (x, t). For the classical harmonic oscillator with Lagrangian,

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}, \tag{1}$$

find values of (x, x_0, t) such that there exists a unique path; no path at all; more than one path. Take $t_0 = 0$, $t_1 = t$ and use τ for a variable intermediate time, $0 \le \tau \le t$, as in the Notes.

(b) Compute Hamilton's principal function $S(x, x_0, t)$ for the harmonic oscillator, and verify the generating function relations,

$$p_1 = \frac{\partial S}{\partial x_1}, \qquad p_0 = -\frac{\partial S}{\partial x_0}, \qquad H = -\frac{\partial S}{\partial t}. \tag{2}$$

Do this for some time t such that there exists only one classical path.

(c) Sakurai (p. 117) says that the classical path is that which minimizes the action. Let $x(\tau)$ be a classical orbit in the harmonic oscillator, satisfying $x(0) = x_0$, $x(t) = x$ for given values of (x, x_0, t). Consider a modified path, $x(\tau) + \delta x(\tau)$, where $\delta x(\tau)$ vanishes at $\tau = 0$ and $\tau = t$. Expand the action along the modified path out to second order in δx, and show that the second variation in the action can be written,

$$\int_0^t d\tau\, \delta x(\tau) B \delta x(\tau), \tag{3}$$

where B is some operator. Find an expression for the operator B, find its eigenvalues β_n and eigenfunctions $\xi_n(\tau)$. Show that if $t < \pi/\omega$, then all eigenvalues are positive, and the action is a minimum. For other values of time t, show that the number of negative eigenvalues of B is the largest integer less than $\omega t/\pi$. Thus, for $t > \pi/\omega$, the classical path is not a minimum of the action functional, but rather a saddle point.

(d) Put the pieces together, and write out the Van Vleck expression for the propagator of the harmonic oscillator, $K(x, x_0, t)$.

(e) Think of the complex time plane, and consider $K(x, x_0, t)$ for times on the real axis satisfying $0 < t < \pi/\omega$. Analytically continue the expression for K in this time interval down onto the negative imaginary time axis, set $t = -i\hbar\beta$, and get an expression for the matrix elements of the Boltzmann operator, $\langle x|e^{-\beta H}|x_0\rangle$ for a harmonic oscillator in thermal equilibrium. Take the trace to get the partition function $Z(\beta)$.

2. The classical Lagrangian for a particle of charge e in a combined magnetic field and scalar potential V is

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m|\dot{\mathbf{x}}|^2}{2} + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) - V(\mathbf{x}). \tag{4}$$

It turns out that the discretized version of the path integral for the corresponding quantum mechanical particle is

$$\begin{aligned} K(\mathbf{x}, \mathbf{x}_0, t) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{3N/2} \int d^3\mathbf{x}_1 \ldots d^3\mathbf{x}_{N-1} \\ \times \exp\Bigl\{\frac{i\epsilon}{\hbar}\sum_{j=1}^{N}\Bigl[\frac{m(\mathbf{x}_j - \mathbf{x}_{j-1})^2}{2\epsilon^2} \\ + \frac{e}{c}\frac{(\mathbf{x}_j - \mathbf{x}_{j-1})}{\epsilon} \cdot \mathbf{A}\Bigl(\frac{\mathbf{x}_j + \mathbf{x}_{j-1}}{2}\Bigr) - V(\mathbf{x}_{j-1})\Bigr]\Bigr\}. \end{aligned} \tag{5}$$

The interesting thing about this path integral is that the vector potential $\mathbf{A}$ is evaluated at the midpoint of the discretized interval $[\mathbf{x}_{j-1}, \mathbf{x}_j]$. Use an analysis like that presented in Sec. 8.14 of the Notes to show that this discretized path integral is equivalent to the Schrödinger equation for a particle in a magnetic field. Show that this would not be so if the vector potential were evaluated at either end of the interval $[\mathbf{x}_{j-1}, \mathbf{x}_j]$ (it must be evaluated at the midpoint). Show that it does not matter which end of the interval the scalar potential is evaluated at.

The delicacy of the points at which the vector potential must be evaluated is related to the fact that the action integrals in the exponent of the Feynman path integral are not really ordinary Riemann sums, because the paths themselves are not differentiable. Instead, they obey the $\Delta x \sim (\Delta t)^{1/2}$ rule discussed in the notes. Casual notation such as Eq. (8.32) glosses over such details.

3. This problem is classical mechanics, but I give it anyway because it might help you gain some insight into magnetic monopoles. Consider the motion of an electron of charge $q = -e$ in the field of a magnetic monopole. Assume the monopole is infinitely massive in comparison to the electron. The monopole produces a field,

$$\mathbf{B}(\mathbf{r}) = g\frac{\mathbf{r}}{|\mathbf{r}|^3}, \tag{6}$$

where g is a constant. Although monopoles have never been observed, nevertheless people have carried out experiments to search for them. In such experiments, it is important to know the behavior of ordinary matter in a monopole field, in order to recognize the signatures a monopole would make in experimental apparatus.

(a) Write down Newton's laws for the electron motion. You may use the abbreviation,

$$\mu = \frac{eg}{mc}. \tag{7}$$

To solve these equations of motion, we begin by a search for constants of motion. If it were an electric monopole instead of a magnetic monopole, then we would obviously have four constants of motion: the energy E and angular momentum $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$. Show that the usual (orbital) angular momentum vector $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$ is not conserved. Show that L^2 is conserved, however. This gives you two time-independent constants of motion, (E, L^2), which are not enough to obtain the complete solution. Therefore we must find more constants of motion.

(b) Consider the vector potential,

$$\mathbf{A} = \frac{-g\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}}. \tag{8}$$

This vector potential differs by a gauge transformation from the vector potentials dis-ussed in class. It is well behaved everywhere except on the z-axis (both positive and negative). Verify that this vector potential gives the magnetic field of Eq. (6). This vector potential is invariant under rotations about the z-axis, that is, the angle ϕ is ignorable. Use this vector potential in a classical Lagrangian in spherical coordinates, and use Noether's theorem to obtain another conserved quantity (in addition to E, L^2). Show that this quantity is L_z plus another quantity which you may call S_z. Write $J_z = L_z + S_z$, so that J_z is the new conserved quantity. As the notation suggests, interpret J_z as the z-component of a vector $\mathbf{J}$, i.e., guess the formula for $\mathbf{J} = \mathbf{L} + \mathbf{S}$ (all 3 components). Hint: to help you with the guess, transform J_z to rectangular coordinates. By resorting to Newton's laws, show explicitly that $\mathbf{J}$ is conserved. You now have four time-independent constants of motion, $(E, \mathbf{J})$. There are only four, because L^2 is a function of $\mathbf{J}$; show this.

(c) By playing around with dot products of various vectors and looking for exact time derivatives, find more constants of motion. Use these to find r^2 as a function of t. Show that the electron can reach the singularity at $r = 0$ only if $L^2 = 0$.

(d) Show that the orbit lies on a cone whose apex is at the origin. Find the opening angle of the cone as a function of $(E, \mathbf{J})$. Find a vector which specifies the axis of the cone. Assume $L^2 \neq 0$, and let $t = 0$ occur at the point of closest approach. Find the distance of closest approach as a function of (E, L).

(e) Choose the axes so that $\mathbf{J}$ is parallel to the z-axis. Find (r, θ, ϕ) as functions of t.

4. Prove the commutation relations (9.30), using Eq. (9.23) and the properties of the Levi-Civita symbol ϵ_{ijk}.