Physics 221A
Fall 2007
Homework 6
Due Friday, October 12, 2007

Reading Assignment: Sakurai, pp. 109–123; Notes 8, through p. 20.

1. The propagator can not only be used for advancing wave functions in time, but also sometimes in space. Consider a beam of particles of energy E in three dimensions launched at a screen in the plane $z = 0$. The particles are launched in the z-direction. The screen has holes in it that allow some particles to go through (for example, it might be a double slit experiement). We will assume that the wave function at $z = 0$ inside the holes is constant, say, $\psi(x, y, 0) = 1$ when (x, y) lies inside a hole, and zero when (x, y) is not in a hole. This is (to within an overall phase) what would happen if we took the plane wave e^{ikz} and just cut it off at the edges of the holes. In other words, $\psi(x, y, 0)$ is the "characteristic function" of the holes. Suppose also that the region $z > 0$ is vacuum. With an extra physical assumption, this information is enough to determine the value of the wave function in the region $z > 0$.

Define a wave number by

$$k_0 = \frac{\sqrt{2mE}}{\hbar}, \tag{1.1}$$

so that $\psi(x, y, z)$ satisfies the wave equation

$$\nabla^2\psi + k_0^2\psi = 0 \tag{1.2}$$

in the region $z > 0$. We would like to solve this wave equation in the region $z > 0$, subject to the given boundary conditions at $z = 0$.

The same equation and boundary conditions also describe some different physics. If plane light waves of a given frequency ω are launched in the z-direction against the screen, and if ψ stands for any component of the electric field, then ψ satisfies Eq. (1.2) with $k_0 = \omega/c$. The problem is one of diffraction theory (either in optics or quantum mechanics).

Write the wave equation (1.2) in the form,

$$-\frac{\partial^2\psi}{\partial z^2} = (k_0^2 + \nabla_\perp^2)\psi, \tag{1.3}$$

where

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.4}$$

(a) Consider now the equation

$$i\frac{\partial\psi}{\partial z} = -\sqrt{k_0^2 + \nabla_\perp^2}\,\psi, \tag{1.5}$$

where the square root of the operator indicated is computed as described in Sec. 1.20 of the Notes. Obviously we have just taken the square roots of the operators appearing on the two sides of Eq. (1.3), with a certain choice of sign. Show that any solution $\psi(x, y, z)$ of Eq. (1.5) is also a solution of Eq. (1.3).

The converse is not true, there are solutions of Eq. (1.3) that are not solutions of Eq. (1.5), but the solutions of Eq. (1.5) all have the property that the waves are travelling in the positive z-direction, something we require on the basis of the physics.

Now suppose that in the region $z > 0$ the angle of propagation of the waves relative to the z-axis is small. This will be the case if the size of the holes in the screen is much larger than a wavelength. Then $\nabla_\perp^2$ acting on ψ is much less than k_0^2 multiplying ψ, so we can expand the square root in Eq. (1.5) to get

$$i\frac{\partial\psi}{\partial z} = -\Big(k_0 + \frac{1}{2k_0}\nabla_\perp^2\Big)\psi. \tag{1.6}$$

This is called the paraxial approximation. Now define a new wave function ϕ by

$$\psi(x, y, z) = e^{ik_0 z}\phi(x, y, z), \tag{1.7}$$

and derive a wave equation for ϕ.

(b) Now write an integral giving $\phi(x, y, z)$ for $z > 0$ in terms of $\phi(x, y, 0)$. Suppose for simplicity there is one hole, and it lies inside the radius $\rho = a$, where

$$\rho = \sqrt{x^2 + y^2}. \tag{1.8}$$

Show that if $z \gg a^2/\lambda$, then $\phi(x, y, z)$ is proportional to the 2-dimensional Fourier transform of the hole (that is, of its characteristic function). This is the Fraunhofer region in diffraction theory. Smaller values of z lie in the Fresnel region, which is more difficult mathematically because the integral is harder to do.

(c) Suppose the hole is a circle of radius a centered on the origin. Evaluate the integral explicitly and obtain an expression for $\psi(x, y, z)$ for z in the Fraunhofer (large z) region. You may find the following identities useful:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{ix\sin\theta}, \tag{1.9}$$

where J_0 is the Bessel function. See Eq. (9.1.18) of Abramowitz and Stegun. Also note the identity,

$$\frac{d}{dx}(xJ_1(x)) = xJ_0(x). \tag{1.10}$$

See Eq. (9.1.27) of Abramowitz and Stegun.

This problem can be used to calculate the forward scattering amplitude in hard sphere scattering, a topic we will take up later.

2. The evaluation of the path integral by the stationary phase approximation requires us to evaluate the determinant of the matrix (8.69), in the limit $N \to \infty$. Fortunately, it can be shown that in this limit, the value of the determinant becomes the solution of a simple differential equation, which has a simple classical interpretation. Furthermore, the result can be expressed in terms of Hamilton's principal function S as shown by Eq. (8.71). This is convenient, because we need S anyway for the phase of the path integral. In this problem we derive Eq. (8.71).

In this problem, as in the lectures and Notes, we denote the final time by t and a variable intermediate time by τ, so that $0 \le \tau \le t$.

(a) In classical mechanics we must often consider the problem of how a small change or error in initial conditions affects the final conditions.

Consider a classical particle in one dimension, moving in potential $V(x)$. The classical equation of motion is

$$m\frac{d^2x(\tau)}{d\tau^2} = -V'(x). \tag{2.1}$$

Suppose the initial conditions are $x(0) = x_0$, $p(0) = p_0$ at $\tau = 0$. The final position at $\tau = t$ is a function of t and of the initial conditions,

$$x(t) = x(x_0, p_0, t). \tag{2.2}$$

Suppose we make small changes δx_0, δp_0 in the initial conditions. Then the change in the position $\delta x(t)$ at the final time is given by

$$\delta x = \left(\frac{\partial x}{\partial x_0}\right)_{p_0} \delta x_0 + \left(\frac{\partial x}{\partial p_0}\right)_{x_0} \delta p_0, \tag{2.3}$$

where we have indicated the variables that are held fixed in the various derivatives.

Assuming that both $x(\tau)$ and $x(\tau) + \delta x(\tau)$ are solutions of Eq. (2.1), and assuming that $\delta x(\tau)$ is small, derive an equation of evolution for $\delta x(\tau)$. This will be expressed in terms of the function $x(\tau)$, which you may take to be given. In other words, you will obtain an equation of evolution for a small perturbation around a given classical orbit.

(b) Use the relations (8.39) to express the coefficient of δp_0 in Eq. (2.3) in terms of Hamilton's principal function S. Compared to the notation of Sec. 8.9, notice that t, t_0 and t_1 of that section become τ, 0 and t here. Also, since the system is time-independent, Hamilton's principal function S depends only on (x, x_0, t).

(c) To apply the stationary phase formula (8.58), we need the determinant of the $(N-1)\times(N-1)$ matrix $Q_{k\ell}$ in Eq. (8.69) in the limit $N\to\infty$. Let D_k be the determinant of the upper left $k\times k$ block of Q, as in the Notes. Show that D_k satisfies Eq. (8.70). It will help to define $D_0 = 1$ and $D_{-1} = 0$. Show that for a free particle, $D_j = j+1$, so $\det Q = N$ in this case. This suggests that even when the potential is nonzero, $\det Q$ diverges as N when $N\to\infty$. As pointed out in the Notes, such a divergence is needed to make the path integral finite. Therefore define

$$F_j = \epsilon D_j, \tag{2.4}$$

so that F_{N-1} will approach a definite limit as $N\to\infty$. Notice that F_j satisfies the same recursion relation as D_j.

(d) Let $\tau_j = j\epsilon$, and let F_j go over to a function $F(\tau)$ in the limit $\epsilon\to 0$. Show that the recursion relation for F_j becomes a differential equation for $F(\tau)$, in fact, the same differential equation you derived in part (a). Also show that $F(0) = 0$ and $F'(0) = 1$.

(e) Use these results to express $F(t)$ in terms of Hamilton's principal function $S(x, x_0, t)$, and prove Eq. (8.81).

This problem shows that the determinant of Q contains information about the paths that are nearby the classical path $x(\tau)$ in path space.