Physics 221A
Fall 2007
Homework 5
Due Friday, October 5, 2007

Reading Assignment: Sakurai, pp. 89–97; Notes 7, lecture notes posted for the week.

1. The eigenfunctions $\psi_n(x)$ of the harmonic oscillator in configuration space are given by Eq. (7.43). In this problem you may use dimensionless units, as in Notes 7. Find the corresponding momentum space wave eigenfunctions, $\phi_n(p)$, which are defined by

$$\phi_n(p) = \int \frac{dx}{\sqrt{2\pi}}\, e^{-ipx}\psi_n(x). \tag{1}$$

Get the phases right.

2. A problem on coherent states.

(a) For a one-dimensional problem, work out the commutators, $[W(z_0), \hat{x}]$ and $[W(z_0), \hat{p}]$, where $W(z_0)$ is the Heisenberg operator, given by Eq. (7.81). Use these to show that the coherent state $|z_0\rangle = W(z_0)|0\rangle$ is an eigenstate of the annihilation operator a with eigenvalue z_0.

(b) The annihilation operator is not Hermitian, so its eigenvalues are not real in general, nor are its eigenstates orthonormal, that is, $\langle z_0|z_1\rangle \neq 0$ when $z_0 \neq z_1$. Nevertheless, they are complete in a certain sense. Prove that

$$\int \frac{dx_0\,dp_0}{2\pi}\, |z_0\rangle\langle z_0| = 1. \tag{2}$$

This resolution of the identity shows that an arbitrary wave function can be expanded as a linear combination of coherent states, that is, Gaussian wave packets. However, the expansion is not unique, because the coherent states are actually overcomplete (they are not linearly independent). In fact, there are many vanishing linear combinations of coherent states (with nonzero coefficients). Nevertheless, the formula (2) looks neat. It can be used as the basis of a path integral.

(c) Use the Heisenberg picture to obtain expressions for $U(t)aU(t)^\dagger$ and $U(t)a^\dagger U(t)^\dagger$, where a and $a^\dagger$ are the usual annihilation and creation operators in the Schrödinger picture, and

$U(t)$ is the time-evolution operator for the harmonic oscillator. Then use these to obtain a simple expression for

$$U(t)W(z_0)U(t)^\dagger. \tag{3}$$

Finally, use that to obtain a simple expression for $U(t)|z_0\rangle$. This approach to finding the time evolution of a coherent state is easier than the one taken in the notes, which involved normal ordered Taylor series.

3. The problem of a charged particle in a uniform magnetic field was discussed in class. We take a negative particle with charge $q = -e$ $(e > 0)$ and ignore the spin (a "spinless electron"). We let $\mathbf{B} = B\hat{\mathbf{z}}$, where B is constant (a uniform magnetic field). As in class, we define the velocity operator,

$$\mathbf{v} = \frac{1}{m}\Bigl(\mathbf{p} + \frac{e}{c}\mathbf{A}\Bigr), \tag{4}$$

where $\mathbf{B} = \nabla\times\mathbf{A}$. We define

$$\omega = \frac{eB}{mc}, \tag{5}$$

the frequency of the classical gyration. We introduce guiding center operators,

$$\begin{aligned} X &= x - v_y/\omega, \\ Y &= y + v_x/\omega. \end{aligned} \tag{6}$$

The Hamiltonian is

$$H = \frac{m}{2}(v_x^2 + v_y^2) + \frac{m}{2}v_z^2 = H_\perp + H_z. \tag{7}$$

Sometimes if we need to distinguish operators from eigenvalues, we will put a hat on the operators, e.g., $\hat{X}$, $\hat{Y}$, $\hat{P}_3$, etc.

The most interesting variables to use for this problem are (X, Y, v_x, v_y, z, v_z), which satisfy the commutation relations,

$$[v_x, v_y] = -\frac{i\hbar\omega}{m}, \qquad [X, Y] = \frac{i\hbar}{m\omega}, \qquad [z, v_z] = \frac{i\hbar}{m}, \tag{8}$$

with all other commutators among these six operators being zero. This leads to the definitions,

$$\begin{aligned} Q_1 &= \sqrt{m\omega}\,X, \qquad & Q_2 &= \sqrt{\frac{m}{\omega}}\,v_y, \qquad & Q_3 &= z, \\ P_1 &= \sqrt{m\omega}\,Y, \qquad & P_2 &= \sqrt{\frac{m}{\omega}}\,v_x, \qquad & P_3 &= mv_z, \end{aligned} \tag{9}$$

which satisfy the canonical commutation relations,

$$[Q_i, P_j] = i\hbar\,\delta_{ij}. \tag{10}$$

The Hamiltonian in terms of the (Q_i, P_i) operators is

$$H = \frac{\omega}{2}(Q_2^2 + P_2^2) + \frac{\hat{P}_3^2}{2m} = H_\perp + H_z. \tag{11}$$

The parallel and perpendicular components of the energy, H_z and $H_\perp$ commute with each other and possess simultaneous eigenstates. It may be slightly better to talk about simultaneous eigenstates of $H_\perp$ and $\hat{P}_3$, which is almost the same since H_z is a function of $\hat{P}_3$. Let n be the quantum number of $H_\perp$ (the usual harmonic oscillator quantum number for the so-called *Landau level*), and let P_3 be the eigenvalue of operator $\hat{P}_3$. Then the energy of a simultaneous eigenstate with these eigenvalues is

$$E_{nP_3} = (n + \tfrac{1}{2})\hbar\omega + \frac{P_3^2}{2m}. \tag{12}$$

To find wave functions we need another operator, commuting with $H_\perp$ and $\hat{P}_3$, to form a complete set of commuting operators. The choice $\hat{X}$ for the third operator will be discussed in class. That choice is useful for making the translational invariance in the y-direction manifest. Here we make a different choice for the third operator, namely,

$$L = L_z = xp_y - yp_x, \tag{13}$$

the angular momentum (z-component only), which makes the rotational invariance about the z-axis manifest.

Equation (13) is the usual definition of angular momentum, but it is only conserved if the vector potential is chosen to be rotationally invariant. We will choose the gauge,

$$\mathbf{A} = \frac{B}{2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}). \tag{14}$$

If you plot the vector field $\mathbf{A}$ in the x-y plane, you will see that it is rotationally invariant.

Since $A_z = 0$, the kinetic momentum in the z-direction $P_3 = mv_z$ is the same as the canonical momentum p_z. Thus we will henceforth write p_z instead of P_3.

(a) Express L in terms of variables (X, Y, v_x, v_y) and then in terms of (Q_1, P_1, Q_2, P_2). Show that it commutes with both $H_\perp$ and $\hat{P}_3$, so that all three operators, $(L, H_\perp, \hat{P}_3)$ commute with one another. Let the eigenvalues of L be written $m\hbar$. Show that m is an integer. For given n (Landau quantum number), what are the allowed values of m?

Actually, a somewhat more convenient operator for the third member of the complete set is

$$K = \frac{1}{2}(Q_1^2 + P_1^2), \tag{15}$$

another harmonic oscillator. Let k be the usual harmonic oscillator quantum number for this oscillator. Express m as a function of n and k. Henceforth let us take $(K, H_\perp, \hat{P}_3)$ as the complete set of commuting observables.

(b) Now we work on finding the wave functions $\psi(x, y, z)$ that are simultaneous eigenfunctions of $(K, H_\perp, \hat{P}_3)$. For the rest of this problem, let us choose units so that $m = \omega = \hbar = 1$, which simplifies the algebra and which entails no loss of physics since physical units can always be restored in the final formulas.

First show that an eigenfunction $\psi(x, y, z)$ of $\hat{p}_z$ with eigenvalue p_z has the form

$$\psi(x, y, z) = \phi(x, y)e^{ip_z z}. \tag{16}$$

This is the easy operator.

Choosing $\phi(x, y)$ so that the function is also an eigenfunction of both K and $H_\perp$ takes more work. We first work on the double ground state $k = 0$, $n = 0$. Introduce annihilation and creation operators a, $a^\dagger$ for oscillator K and another pair b, $b^\dagger$ for oscillator $H_\perp$. The double ground state wave function must be annihilated by both a and b. That is, we want

$$(a\phi_{00})(x, y) = (b\phi_{00})(x, y) = 0. \tag{17}$$

Write out a and b as differential operators in x and y.

The subsequent analysis is simplified if we use complex coordinates w, $\bar{w}$ in the x-y plane, defined by

$$\begin{aligned} w &= \frac{1}{\sqrt{2}}(x + iy), \\ \bar{w} &= \frac{1}{\sqrt{2}}(x - iy), \end{aligned} \tag{18}$$

so that

$$\begin{aligned} \frac{\partial}{\partial w} &= \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \\ \frac{\partial}{\partial \bar{w}} &= \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \end{aligned} \tag{19}$$

Transform the differential operators for a and b to the w, $\bar{w}$ coordinates, and solve Eq. (17) for $\phi_{00}(x, y)$. Normalize it.

(c) Now write out differential operators in the w, $\bar{w}$ coordinates for the creation operators $a^\dagger$ and $b^\dagger$. Use them to obtain the normalized wave function ϕ_{kn} for $k = 0$, $n > 0$.

(d) Use the Rodriguez formula for the associated Laguerre polynomials,

$$L_n^m(x) = \frac{1}{n!} e^x x^{-m} \frac{d^n}{dx^n} (x^{n+m} e^{-x}) \tag{20}$$

[see Abramowitz and Stegun, Eq. (22.11.6)], to express the normalized wavefunction $\phi_{kn}(x, y)$ for any values of k and n in terms of associated Laguerre polynomials. Hint: Use the identity,

$$\left(\frac{\partial}{\partial w} - \frac{\bar{w}}{2}\right) e^{w\bar{w}/2} f(w, \bar{w}) = e^{w\bar{w}/2} \frac{\partial}{\partial w} f(w, \bar{w}), \tag{21}$$

where f is any function.