Physics 221A
Fall 2007
Homework 3
Due Thursday, September 20, 2007

Reading Assignment: Sakurai, pp. 44-60, 174–187, the rest of Notes 4, Notes 5, other lecture notes posted for the week on the web site.

1. In quantum mechanics the measurement of one observable introduces an uncontrollable and unpredictable disturbance in the value of any observable that does not commute with the one being measured, as discussed in Sec. 2.6 of the Notes. For example, a measurement of μ_z in a Stern-Gerlach apparatus causes the values of μ_x and μ_y to become completely undetermined.

We can understand how this disturbance comes about in a classical model. To measure μ_z, we must use a magnetic field in the z-direction, but this causes μ_x and μ_y to precess, so their values on emerging from the Stern-Gerlach apparatus are different from their values when they entered. If classical mechanics were valid, we could calculate the precession angle for any particular particle in the beam, and compensate for the evolution in μ_x and μ_y. The beam has some spatial extent, however, so particles do not follow the same trajectory and their spins do not precess by the same angle, but if the size of the beam is made small enough the spread in these angles can be made as small as we like. In particular, it can be made $\ll 2\pi$, giving us a definite phase angle for the whole beam, and therefore known (and controllable) effects on μ_x and μ_y when we measure μ_z.

Show that if we try to do this in quantum mechanics, we wash out the effect we are trying to observe, namely, the splitting of the beam into two beams that make two spots on a screen. For simplicity you may assume that Eq. (2.37) is valid.

2. Measuring the Schrödinger wave function is not like measuring a classical field, such as an electric field. Consider a scattering gedankenexperiment in which spinless particles of a given energy are directed at a target, as in the figure. We wish to measure the wave function downstream from the scatter. We assume the beam is described by a pure state, and that the incident wave is a plane wave (over a sufficiently large spatial region). The beam is low density, so the particles do not interact with one another. To measure $|\psi(\mathbf{r})|^2$ over some volume of space, we just put a screen S in a certain location, and gather enough statistics to get the probability density on this surface. We then move the surface and measure again.

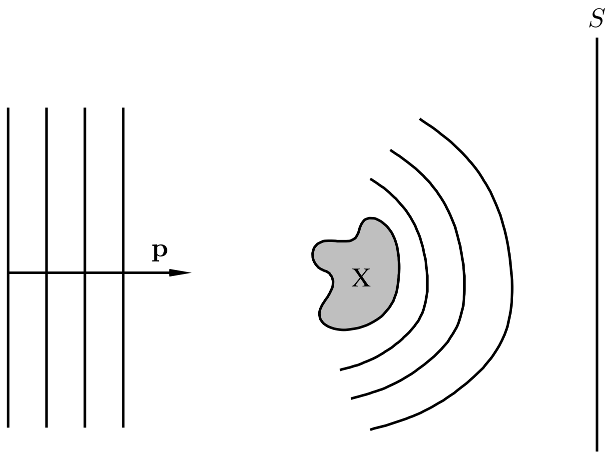

Figure for problem 2. A scattering experiment.

Describe a modification to this gedankenexperiment by which the phase of the wave function $\psi(\mathbf{r})$ can be measured on the screen, apart from the overall phase, which of course is nonphysical and can never be measured.

3. Let $|\psi\rangle$ be the state of a spinless particle in three dimensions, and let $\phi(\mathbf{p}) = \langle\mathbf{p}|\psi\rangle$ be its momentum space wave function. Find the momentum space wave function of the state $T(\mathbf{a})|\psi\rangle$, that is, find the action of the translation operator $T(\mathbf{a})$ in the momentum representation.

4. In this problem we denote operators with a hat, as in $\hat{x}$ or $\hat{A}$, and we denote eigenvalues or classical quantities without a hat, as in x or $A(x,p)$. We work in one dimension, and think of a wave function $\psi(x)$ or $\psi(x,t)$.

If $\hat{A}$ is an operator, we define the *Weyl transform* of $\hat{A}$, denoted $A(x,p)$, by

$$A(x,p) = \int_{-\infty}^{+\infty} ds\, e^{-ips/\hbar}\, \langle x + s/2|\hat{A}|x - s/2\rangle. \tag{4}$$

Here the notation $|x - s/2\rangle$, for example, means the eigenket of $\hat{x}$ with eigenvalue $x - s/2$. It is useful to think of $A(x,p)$ as a function defined on the classical (x,p) phase space which is in some sense the classical observable corresponding to the quantum operator $\hat{A}$.

(a) Show that if $A(x,p)$ is the Weyl transform of operator $\hat{A}$, then $A(x,p)^*$ is the Weyl transform of $\hat{A}^\dagger$. In particular, this shows that the Weyl transform of a Hermitian operator is a real function on phase space.

(b) Show that if operators $\hat{A}$ and $\hat{B}$ have Weyl transforms $A(x,p)$ and $B(x,p)$, respectively, then

$$\mathrm{tr}(\hat{A}^\dagger \hat{B}) = \int \frac{dx\,dp}{2\pi\hbar}\, A(x,p)^* B(x,p). \tag{5}$$

Notice how the right hand side looks like the "scalar product" of two classical observables on phase space.

(c) Find the Weyl transforms of the following operators: 1 (the identity operator); $\hat{x}$; $\hat{p}$; $\hat{x}\hat{p}$; $\hat{p}\hat{x}$; $\hat{p}^2/2m + V(\hat{x})$.

(d) Let $W(x,p)$ be the Weyl transform of the density operator $\hat{\rho}$. Since $\hat{\rho}$ is Hermitian, $W(x,p)$ is real. Interpret the integrals

$$\int_{-\infty}^{\infty} dx\, W(x,p) \qquad \text{and} \qquad \int_{-\infty}^{\infty} dp\, W(x,p), \tag{6}$$

physically and compare to the corresponding integrals of $\rho(x,p)$ in classical statistical mechanics, where ρ is the classical probability density in phase space.

Now some comments. These results suggest that $W(x,p)$ is a distribution function of particles in phase space whose statistics reproduces the statistics inherent in quantum measurement. Unlike a classical distribution function $\rho(x,p)$, however, $W(x,p)$ can take on negative values. These "negative probabilities" have no meaning in any statistical sense, but they only arise, in a certain sense, when we attempt to measure x and p simultaneously to a precision greater than that allowed by the uncertainty principle.

5. Consider the Hamiltonian for a particle of charge q in an electromagnetic field:

$$H = \frac{1}{2m}\Big[\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{x},t)\Big]^2 + q\Phi(\mathbf{x},t), \tag{7}$$

where we allow all fields to be space- and time-dependent.

(a) Define the kinetic momentum operator $\boldsymbol{\pi}$ by

$$\boldsymbol{\pi} = \mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{x},t). \tag{8}$$

Notice that $\boldsymbol{\pi}$ has an explicit time dependence, due to the $\mathbf{A}$ term. Work out the commutation relations, $[x_i, \pi_j]$ and $[\pi_i, \pi_j]$. Use these to work out the Heisenberg equations of motion for the operators $\mathbf{x}$, $\boldsymbol{\pi}$. Then eliminate $\boldsymbol{\pi}$, and find an expression for $\ddot{\mathbf{x}}$ (the Heisenberg analog of the Newton-Lorentz equations).

(b) By taking expectation values, show that if $\mathbf{B}$ is uniform in space and $\mathbf{E}$ has the form,

$$E_i(\mathbf{x}) = a_i + \sum_j b_{ij}\, x_j, \tag{9}$$

where a_i and b_{ij} are constants, then the expectation value of $\mathbf{x}$ follows the classical orbit.