**Physics 221B**
**Spring 2008**
**Homework 26**
**Due Friday, April 25 at 5pm**

**Reading Assignment:** Lecture notes for 4/15/08, 4/17/08; Sakurai, *Advanced Quantum Mechanics*, pp. 95–99; Reprint on Covariance of the Dirac Equation (chapter 2 of Bjorken and Drell), through p. 21 or so.

**1.** The Dirac equation in two space dimensions. Suppose we lived in a world with two space dimensions ($x$ and $y$) and one time dimension. Let $x^\mu = (ct, x, y)$, and otherwise use the obvious restrictions of ordinary relativity theory to two spatial dimensions (for example, $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1)$). In two spatial dimensions, the vector potential $\mathbf{A}$ has two components $(A_x, A_y)$, but the magnetic field is a scalar,

$$B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \tag{1}$$

More generally, true vectors in three dimensions become 2-vectors when restricted to two dimensions, but pseudo-vectors in three dimensions become (pseudo) scalars in two dimensions. You will find that in some respects the Dirac equation in two spatial dimensions is very similar to what was done in lecture in three spatial dimensions, and in other respects it is different.

**(a)** Carry out Dirac's program of finding a relativistic wave equation that is first order in both space and time. Show that the Dirac algebra of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\beta$ matrices can be satisfied by $2 \times 2$ Hermitian matrices. This is the simplest solution for the Dirac algebra in $2+1$ dimensional space-time. Thus, the Dirac spinor has two components. The solution is not unique, because you can change the basis (conjugate any solution with some unitary matrix to create another representation of the algebra), so let your primary solution be one in which $\beta$ is diagonal. Call this the "Dirac-Pauli" representation. Also compute the matrices $\gamma^\mu$ in the Dirac-Pauli representation. The Dirac-Pauli representation is most convenient for studying the nonrelativistic limit.

Now find a representation in which the matrices $\gamma^\mu$ are purely imaginary. Call this the "Maiorana representation." The Maiorana representation makes Lorentz transformations on the spinors look most simple.

In the remaining parts of this problem, carry out your calculations in a representation independent manner, that is, using only the algebraic properties of the matrices in question, unless the problem calls for a specific representation (for example, part (d)).

**(b)** Write out the Dirac equation including minimal coupling to the electromagnetic field, and show that it possesses a positive definite probability density with an associated probability current that taken together satisfy the continuity equation.

**(c)** Work out the Heisenberg equations of motion for $\mathbf{x}$ and $\boldsymbol{\pi} = \mathbf{p} - (q/c)\mathbf{A}$.

**(d)** Using the Dirac-Pauli representation, write the 2-component Dirac spinor as

$$\psi = e^{-imc^2t/\hbar}\begin{pmatrix}\phi \\ \chi\end{pmatrix}, \tag{2}$$

and assume that the energy is $E = mc^2 + \text{small}$. Find an approximation giving $\chi$ in terms of $\phi$, and use it to obtain an effective Schrödinger equation for the upper component $\phi$. Is the resulting Schrödinger equation realistic for any problems in our real (3-dimensional) world?

**(e)** Assume that the 2-component Dirac wave function transforms under proper Lorentz transformations $\Lambda$ in $2+1$ dimensions according to

$$\psi'(x) = D(\Lambda)\psi(\Lambda^{-1}x), \tag{3}$$

where $D(\Lambda)$ is some (as yet unknown) $2\times 2$ representation of the proper Lorentz transformations in $2+1$ dimensions and $x = (ct, x_1, x_2)$ (1,2 mean $x$, $y$). Assuming that $\psi(x)$ satisfies the free particle Dirac equation, and that $\psi'(x)$ is given by Eq. (3), demand that $\psi'(x)$ also satisfy the free particle Dirac equation and thereby derive a condition that the representation $D(\Lambda)$ must satisfy.

**(f)** Write out explicitly the matrices $D(\Lambda)$ for the case of pure rotations and pure boosts. Do this in the Dirac-Pauli representation. Show that if you work in the Maiorana representation, the $D$-matrices are purely real.