Physics 221B
Spring 2008
Homework 25
Due Friday, April 18 at 5pm

Reading Assignment: Lecture notes for 4/1/08, 4/3/08, 4/8/08, 4/10/08; Sakurai, *Advanced Quantum Mechanics*, pp. 53–57, 64–72.

Sakurai uses the complex coordinate $x_4 = ict$ in his presentation of special relativity. This has the advantage that the Minkowski metric becomes Euclidean, and it is not necessary to distinguish between contravariant and covariant indices. This was the frequently the custom in the older literature on special relativity and the Dirac equation. On the other hand, this approach is unphysical (time is real, not imaginary).

In my lectures I have been following the presentation of Bjorken and Drell, *Relativisitic Quantum Mechanics*, on the Dirac equation. They use real time but the non-Euclidean or Minkowski metric $g_{\mu\nu}$. This is definitely more in fashion nowadays, especially since there is wider interest in general relativity where one must use a non-Euclidean metric. Nevertheless, I will ask you to read over Sakurai's presentation, because his discussion of the physics is good. You should be able to follow in outline what Sakurai does with the Dirac equation, without going into the details of his complex coordinates. With that in mind, please read Sakurai, pp. 75–91.

1. Calculate the total cross section σ in cm^2 for the elastic scattering of a resonant photon from a hydrogen atom in its ground state $|G\rangle = |n\ell j f m_f\rangle = |10\frac{1}{2}00\rangle$. The photon has the 21 cm wavelength (it is exactly on resonance). Make reasonable approximations.

Hints: Notice that since the ground state is a state of total angular momentum zero, the cross section cannot depend on the polarization of the incident photon. Use atomic units in the calculation, then convert to ordinary units at the end (it's much easier). Do the sum on m_f first. Notice that in the space spanned by $|m_s\rangle|m_i\rangle$, where $m_s, m_i = \pm\frac{1}{2}$, the states $|fm_f\rangle$ form a resolution of the identity, so that

$$\sum_{fm_f} |fm_f\rangle\langle fm_f| = |00\rangle\langle 00| + \sum_{m_f} |1m_f\rangle\langle 1m_f| = 1. \tag{1}$$

The frequency of the resonant photon (in a.u.) is

$$\omega = \frac{4}{3}\frac{g}{m_p}\alpha^2, \tag{2}$$

where m_p is the mass of the proton and g is its g-factor. The decay rate of the $f = 1$ state (in a.u.) is

$$\Gamma = \frac{2^6}{3^4} \frac{g^3 \alpha^{11}}{m_p^3}, \tag{3}$$

at least this is the answer I got to the earlier homework problem, correct it and let me know if you think it is wrong.

Another hint is that when you integrate over solid angles (the solid angle of some wave vector $\mathbf{k}$), you can use the identity,

$$\int d\Omega_{\mathbf{k}}\, k_i k_j = \frac{4\pi}{3} k^2\, \delta_{ij}. \tag{4}$$

You can prove this just by writing out the components k_i of $\mathbf{k}$ in spherical coordinates. It's also related to the Clebsch-Gordan product $1 \otimes 1 = 0 \oplus 1 \oplus 2$, in which you're picking out the $\ell = 0$ part of the product.

2. Sakurai, problem 2-11, p. 74.