Physics 221B
Spring 2008
Homework 22
Due Friday, March 14 at 5pm

Reading Assignment: Rest of Notes 34, Notes 36; Sakurai, Advanced Quantum Mechanics, pp. 20–32. Sakurai uses Heaviside-Lorentz units, which are the same as Gaussian units except the $4\pi$'s have been moved around. See Appendix A.

Notes 35 are optional. They are only if you want to see how the Hamiltonian for the classical electromagnetic field is derived systematically (without guessing), that is, by passing through a Lagrangian. Some students do. Also discussed in those notes are the derivation of the conserved quantities of the matter-field system (energy, momentum, angular momentum), using Noether’s theorem.

1. Several guesses were made in deriving the classical Hamiltonian for the matter-field system, Eqs. (34.95)–(34.97), so we should check the answer to see that it give the correct equations of motion. This Hamiltonian is a function of the $(x_\alpha, p_\alpha)$ and the $(Q_\lambda, P_\lambda)$, which are the $q$’s and $p$’s of the system. All dynamical variables are considered to be functions of the $q$’s and $p$’s. For example, the vector potential is defined by

$$A(r) = \sqrt{\frac{2\pi\hbar}{V}} \sum_\lambda \frac{1}{\sqrt{\omega}} (a_\lambda \epsilon e^{i\mathbf{k}\cdot\mathbf{r}} + c.c.),$$

where

$$a_\lambda = \frac{Q_\lambda + iP_\lambda}{\sqrt{2\hbar}}.$$  \hspace{1cm} (2)

The quantity $a_\lambda$ is classical, in spite of the $\hbar$. We could have written $A$ in terms of the mode amplitude $C_\lambda$, defined in class and in the notes, but $a_\lambda$, which is proportional to $C_\lambda$, is more convenient since it carries over to the annihilation operator in quantum mechanics.

(a) Show that Hamilton’s classical equations for $Q_\lambda$ and $P_\lambda$ are equivalent to

$$\dot{a}_\lambda = -\frac{i}{\hbar} \frac{\partial H}{\partial a_\lambda^*},$$

$$\dot{a}_\lambda^* = \frac{i}{\hbar} \frac{\partial H}{\partial a_\lambda},$$

which are more convenient than the usual ones in $Q_\lambda$ and $P_\lambda$. Notice that these equations are complex conjugates of each other.
(b) What we mean by \( \partial A(r)/\partial t \) is the time derivative of the field \( A \), evaluated at some field point \( r \), due to the time evolution of the \( a_\lambda \) and \( a_\lambda^* \). We also call this \( \dot{A}(r) \). Show that Hamilton’s equations imply

\[
\dot{A}(r) = \sqrt{\frac{2\pi\hbar e^2}{V}} \sum_\lambda \sqrt{\omega_\lambda} (-ia_\lambda \epsilon e^{ik \cdot r} + \text{c.c.}).
\]  

(4)

Hint: Notice the resolution of identity, Eq. (34.53).

(c) Now show that

\[
\ddot{A} = c^2 \nabla^2 A + 4\pi c J_\perp.
\]  

(5)

This equation is equivalent to Maxwell’s equations. Also show that

\[
m_\alpha \ddot{x}_\alpha = q_\alpha \left[ E(x_\alpha) + \frac{1}{c} \dot{x}_\alpha \times B(x_\alpha) \right].
\]  

(6)

2. The momentum operator for the quantized electromagnetic field plus matter is

\[
P = \sum_\lambda \hbar k a_\lambda^\dagger a_\lambda + \sum_\alpha p_\alpha.
\]  

(7)

Let \( A(r) \) be the quantum field,

\[
A(r) = \sqrt{\frac{2\pi\hbar e^2}{V}} \sum_\lambda \frac{1}{\sqrt{\omega_k}} \left( \epsilon_\lambda a_\lambda e^{ik \cdot r} + \text{h.c.} \right).
\]  

(8)

Let

\[
T(b) = \exp \left( -\frac{i b \cdot P}{\hbar} \right),
\]  

(9)

where \( b \) is a constant vector. Compute

\[
T(b)A(r)T(b)^\dagger.
\]  

(10)

Also compute

\[
T(b)x_\alpha T(b)^\dagger,
\]  

\[
T(b)p_\alpha T(b)^\dagger.
\]  

(11)