Physics 221B
Spring 2008
Homework 21
Due Friday, March 7 at 5pm

Reading Assignment: Rest of Notes 31, lecture notes for 2/26/08 on Møller wave operator, Born series and relations between unperturbed and exact Green's operators. The part on the optical theorem is optional, since I did not lecture on it. Also please read the beginning of Notes 34.

Two sets of optional notes have been posted. Notes 32 concerns the transiton matrix, which is also discussed in Sakurai, but I skipped it in lecture for lack of time. It is used in scattering theory to determine the signatures of various conservation laws (or their violation) in scattering experiments. Some of the material in Notes 32 repeats some of the material in the lecture notes for 2/26/08. Notes 33 contains material on adiabatic invariance and the Born-Oppenheimer approximation, which I have also skipped for lack of time. We may return to this later if there is time.

1. This problem is a variation on Sakurai, problem 7.1.

(a) Find the free-particle Green's function in one dimension,

$$G_{0+}(x,x';E) = \lim_{\epsilon\to 0}\langle x|\frac{1}{E+i\epsilon-H_0}|x'\rangle, \tag{1}$$

where

$$H_0 = \frac{p^2}{2m}. \tag{2}$$

Do this for both $E>0$ and $E<0$, as in the 3-dimensional case discussed in Notes 31. Also find $G_{0-}(x,x';E)$. Show explicitly that they satisfy

$$\Big(E+\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Big)G_{0\pm}(x,x';E) = \delta(x-x'). \tag{3}$$

(b) Write down a 1-dimensional version of the Lippmann-Schwinger equation for an exact scattering solution $\psi(x)$ associated with an incoming (from the left) free particle state $\phi(x) = e^{ikx}/\sqrt{2\pi}$. The exact solution $\psi(x)$ satisfies the Schrödinger equation in a potential $V(x)$, which you can consider to be localized. Consider asymptotic forms (large $|x|$) and find expressions for the transmission and reflection amplitudes t and r which are analogous to Eq. (31.95) in three dimensions. These amplitudes are defined by

$$\psi(x) = \frac{1}{\sqrt{2\pi}}[e^{ikx} + re^{-ikx}], \qquad (x \to -\infty),$$
$$\psi(x) = \frac{1}{\sqrt{2\pi}}te^{ikx}, \qquad (x \to +\infty). \tag{4}$$

(c) Consider the potential,

$$V(x) = \lambda\delta(x). \tag{5}$$

This potential can be seen as the limit of a rectangular barrier (for $\lambda > 0$) with width a and height $V_0 = \lambda/a$ as $a \to 0$. The attractive case $\lambda < 0$ is similar.

Solve the Lippmann-Schwinger equation directly, and write out explicit forms for the wave function $\psi(x)$ for $x < 0$ and $x > 0$. To help the reader, please use the abbreviation,

$$D = \frac{m\lambda}{\hbar^2 k}, \tag{6}$$

as much as possible. Note that D is dimensionless. Compute t and r in terms of D and show explicitly that $|t|^2 + |r|^2 = 1$.

(d) The operator equation,

$$G_+(E) = G_{0+}(E) + G_{0+}(E)VG_+(E), \tag{7}$$

was proved in lecture. It is a kind of Lippmann-Schwinger equation for the exact Green's function. Write this out as an integral equation for the exact Green's function, assume the potential is given by (5), and solve for $G_+(x, x'; E)$. Consider the case $\lambda < 0$. Show that this Green's function has one pole on the negative energy axis, located at the energy of the one (and only) bound state. Show that the residue of this pole is the projection operator onto the eigenspace of this bound state.

2. Sakurai, problem 7.2.

3. Consider N static spherically symmetric scattering centers placed on a straight line such that the n-th scatterer is at point $(n-1)\mathbf{a}$, for $n = 1, \ldots, N$. A particle with incident momentum $\hbar\mathbf{k}$ such that $\mathbf{k} \cdot \mathbf{a} = 0$ is scattered from the array. Assuming the validity of the Born approximation, show that the elastic differential cross section is of the form,

$$\frac{d\sigma}{d\Omega} = |F(\alpha)|^2 \frac{d\sigma_0}{d\Omega}, \tag{8}$$

where $d\sigma_0/d\Omega$ is the differential cross section for scattering by a single scatterer, where α is the angle between $\mathbf{a}$ and $\mathbf{k}'$. Find the form factor $F(\alpha)$.

4. Retardation in the Coulomb gauge. A particle with charge q is located at the origin of the coordinates. In the interval 0 to T the particle is displaced from the origin to $\mathbf{x}(T)$ along a path $\mathbf{x}(t)$ $(0 \le t \le T)$. Let $\mathbf{r}$ be a point distant from the origin, $r \gg |\mathbf{x}(t)|, cT$. The purpose of this exercise is to prove, starting with Maxwell's equations, that the instantaneous variations of the longitudinal electric field created by charge q at $\mathbf{x}$ are exactly compensated by the instantaneous component of the transverse electric field produced by the displacement of the particle.

(a) Calculate, as a function of $\mathbf{x}(t)$, the electric field $\mathbf{E}_{\|}(\mathbf{r},t)$ at point $\mathbf{r}$ and time t from charge q. Show that $\mathbf{E}_{\|}(\mathbf{r},t)$ can be written,

$$\mathbf{E}_{\|}(\mathbf{r},t) = \mathbf{E}_{\|}(\mathbf{r},0) + \delta\mathbf{E}_{\|}(\mathbf{r},t), \tag{9}$$

where $\delta\mathbf{E}_{\|}$ is given by a power series in $|\mathbf{x}(t)|/r$. Show that the lowest order term of this expansion can be expressed as a function of $q\mathbf{x}(t)$ and of the transverse δ-function, $\Delta^{\perp}_{ij}(\mathbf{r})$.

(b) Find the current $\mathbf{J}(\mathbf{r},t)$ associated with the motion of the particle. Express the transverse current $\mathbf{J}_{\perp}(\mathbf{r},t)$ at the point of observation $\mathbf{r}$ as a function of $q\dot{\mathbf{x}}(t)$ and the transverse δ-function $\Delta^{\perp}_{ij}(\mathbf{r}-\mathbf{x}(t))$. Show that to the lowest order in $|\mathbf{x}(t)|/r$, one can replace $\Delta^{\perp}_{ij}(\mathbf{r}-\mathbf{x}(t))$ by $\Delta^{\perp}_{ij}(\mathbf{r})$. Write the Maxwell equation giving $\partial\mathbf{E}_{\perp}(\mathbf{r},t)/\partial t$ as a function of $\mathbf{J}_{\perp}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. Begin by ignoring the contribution of $\mathbf{B}$ to $\partial\mathbf{E}_{\perp}/\partial t$. Integrate the equation between 0 and t. Show that the transverse electric field $\mathbf{E}_{\perp}(\mathbf{r},t)$ produced by $\mathbf{J}_{\perp}(\mathbf{r},t)$ compensates exactly (to lowest order in $|\mathbf{x}(t)|/r$) the field $\delta\mathbf{E}_{\|}(\mathbf{r},t)$ found in part (a). The small parameter here is $|\mathbf{x}|/r$, not v/c; the particle motion could be fast (relativistic).

(c) By eliminating the transverse electric field between the Maxwell equations for the transverse fields, find the equation of motion for the magnetic field $\mathbf{B}$. Show that the source term in this equation can be written in a form which only involves the total current $\mathbf{J}$. Justify the approximation made above of neglecting the contribution of $\mathbf{B}$ to $\partial\mathbf{E}_{\perp}/\partial t$ over short periods ($T \ll r/c$).