Physics 221A
Fall 2007
Homework 2
Due Thursday, September 13, 2007

Reading Assignment: Sakurai, pp. 41–43, the rest of Notes 2, Notes 3, the beginning of Notes 4.

1. Let A and B be observables, for simplicity with discrete spectra.

(a) Show that $[A, B] = 0$ if and only if $[P_{An}, P_{Bm}] = 0$ for all n, m, where P_{An} and P_{Bm} are projectors defined below Eq. (2.25).

(b) Show that the probabilities defined in Eqs. (2.29) and (2.30) are equal for all n, m and $|\psi_0\rangle$ if and only if $[A, B] = 0$.

2. This problem arose out of the thought that maybe the last, unknown sign multiplying μ_y in Eq. (2.23) could be determined if we investigated tandem Stern-Gerlach measurements at all possible orientations, not just at right angles.

In this problem we imagine that we know nothing about Pauli matrices or the formalism of spin, but we do know the measurement postulates of quantum mechanics, and we do know certain experimental facts.

In particular, suppose we have a collimated beam of silver atoms from an oven which is passed through a Stern-Gerlach apparatus oriented in the $\hat{\mathbf{n}}_1$ direction. The beam splits in two, and the beam corresponding to spin pointing in the negative $\hat{\mathbf{n}}_1$ direction is thrown away, while the beam corresponding to spin pointing in the positive $\hat{\mathbf{n}}_1$ direction is passed onto a second Stern-Gerlach apparatus, oriented in the $\hat{\mathbf{n}}_2$ direction. Experimentally (or perhaps we should say, gedanken-experimentally), it is found that the probability that an atom entering the second apparatus will be measured to point in the positive $\hat{\mathbf{n}}_2$ direction is given by

$$\frac{1}{2}(1 + \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) = \cos^2 \gamma/2, \tag{1}$$

where γ is the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$,

$$\cos \gamma = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2. \tag{2}$$

We take γ to lie in the range $0 \le \gamma \le \pi$. The probability of measuring the spin to lie in the negative $\hat{\mathbf{n}}_2$ direction is complementary to (1), namely,

$$\frac{1}{2}(1 - \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) = \sin^2 \gamma/2. \tag{3}$$

We notice that all spin measurements produce only two possible results, so we take the Hilbert space of our quantum system to be 2-dimensional, so that the eigenspaces of the operators which corresponds to measurements of spin are nondegenerate. We let $S(\hat{\mathbf{n}})$ or $S(\theta, \phi)$ be the operator corresponding to measuring spin with an apparatus pointing in the $\hat{\mathbf{n}} = (\theta, \phi)$ direction, and we let $|S(\hat{\mathbf{n}})\pm\rangle$ or $|S(\theta, \phi)\pm\rangle$ be normalized eigenkets corresponding to the component of spin in the $\hat{\mathbf{n}}$ direction taking on the values $\pm\hbar/2$. These eigenkets are subject to phase conventions which we will have to specify further.

(a) At first we are interested in expressing the kets $|S(\theta, \phi)\pm\rangle$ in terms of some standard eigenkets of the operator $S_z = S(\hat{\mathbf{z}}) = S(0, 0)$, which we call $|+\rangle$ and $|-\rangle$. Therefore we consider the case that $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}} = (\theta, \phi)$, and $\hat{\mathbf{n}}_2 = \hat{\mathbf{z}}$. Write the kets $|S(\theta, \phi)\pm\rangle$ as linear combinations of the kets $|\pm\rangle$ with unknown coefficients. Show that by using the experimental results (1) and (3), and phase conventions for the kets $|S(\theta, \phi)\pm\rangle$, it is possible to make the coefficient of $|+\rangle$ real and nonnegative in the expression for $|S(\theta, \phi)+\rangle$, and the coefficient of $|-\rangle$ real and nonnegative in the expression for $|S(\theta, \phi)-\rangle$, and to determine all coefficients apart from a single phase factor, say, $e^{i\beta}$, where β is an as-yet-unknown function of (θ, ϕ). Show that with these phase conventions, we have $|S(0, 0)+\rangle = |+\rangle$, and $|S(0, 0)-\rangle = |-\rangle$.

(b) Now write out the operators $S(\theta, \phi)$ in terms of the basis operators $|+\rangle\langle+|$, $|+\rangle\langle-|$, etc., and in terms of the unknown phase β. Show that by changing the phase conventions for all kets $|S(\theta, \phi)+\rangle$, including $|+\rangle$, by a constant phase factor, independent of (θ, ϕ), and similarly by changing all kets $|S(\theta, \phi)-\rangle$, including $|-\rangle$, by a (possibly different) constant phase factor, it is possible to make $\beta(\theta, \phi)$ vanish at one chosen value of (θ, ϕ). Do this for $(\theta, \phi) = (\pi/2, 0)$, so that $S_x = S(\pi/2, 0)$ will have real matrix elements in the $|\pm\rangle$ basis.

(c) Now return to the case of arbitrary $\hat{\mathbf{n}}_1 = (\theta_1, \phi_1)$ and $\hat{\mathbf{n}}_2 = (\theta_2, \phi_2)$, and try to use the experimental data (1) and (3) to determine the function $\beta(\theta, \phi)$. Is this function completely determined by the experimental data, or is there still some ambiguity?

Finally, prove the formula,

$$S(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \mathbf{S}. \tag{4}$$

(d) Were you able to find the last, unknown sign in μ_y in Eq. (2.23)? If not, how do you think Pauli knew what sign to choose?

3. The projection postulate of quantum mechanics says that if a system is described by a pure state $|\psi\rangle$ (here assumed to be normalized), then after a measurement of the operator A producing eigenvalue a_n, the system is described by the (normalized) state

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}, \tag{5}$$

where P_n is the projector onto the n-th eigenspace of A.

Suppose instead the system is described initially by a density operator ρ (assumed normalized). What is the (normalized) density operator ρ' after the measurement? Express your answer in terms of the original density operator ρ. Do not assume the eigenvalue a_n is nondegenerate.

4. Show that if ρ is a density operator, then $\operatorname{tr}\rho^2 \le 1$, with equality if and only if ρ represents a pure state.

5. Here is another approach to the density operator. Suppose we have a total system t which is composed of a subsystem s of interest, plus the rest r of the total system. For example, let system s be an atom. Then system r could be another microscopic system which has interacted with the atom; it could be a second, identical atom, with both atoms flying apart and carrying spin correlations as in an Einstein-Podolsky-Rosen experiment; it could be a heat bath; it could be a macroscopic measuring apparatus; it could be the rest of the universe. System r may interact with system s only for certain time intervals, as we wish (in other words, we may wish to assume that systems s and r are noninteracting at certain times).

Let $|n\rangle^{(s)}$ and $|m\rangle^{(r)}$ be bases of kets in the Hilbert spaces for the subsystem of interest and the rest of the system, respectively, so that the product kets $|n\rangle^{(s)}|m\rangle^{(r)}$ are basis kets for the Hilbert space for the total system. Let spaces $\mathcal{E}_s$, $\mathcal{E}_r$ and $\mathcal{E}_t$ be the Hilbert spaces for the system of interest, the rest of the system, and the total system, respectively.

Suppose the total system is in a pure state $|\psi\rangle \in \mathcal{E}_t$. If systems s and r are noninteracting (at some time), you might suppose that we could describe system s by means of a wave function, i.e., as a pure state; but this is impossible in general. Show that if A_s is an operator which acts only on ket space $\mathcal{E}_s$, then the expectation value of A_s can be expressed in terms of an operator ρ_s which also acts only on $\mathcal{E}_s$. That is, show that with a proper definition of ρ_s, we have

$$\langle A_s\rangle = \operatorname{tr}(\rho_s A_s). \tag{6}$$

Express ρ_s in terms of the expansion coefficients of $|\psi\rangle$ with respect to a basis in $\mathcal{E}_t$. Show that ρ_s satisfies all the requirements of a density operator, namely, it is Hermitian, nonnegative definite, and has unit trace.