Physics 221B
Spring 2008
Homework 19
Due Friday, February 22 at 5pm

Reading Assignment: Sakurai, pp. 316–335, 399–416; Notes 29, lecture notes for 2/14/08. I am also posting Notes 30, on the photoelectric effect, but it is optional reading. Those notes deal with the photoelectric effect, as an application of time-dependent perturbation theory.

1. Some questions involving the scattering of identical particles. To do this problem you need know only the material in Notes 29, which includes some applications to scattering theory.

(a) In classical mechanics we can always distinguish particles by placing little spots of paint on them. Suppose we have two particles in classical mechanics which are identical apart from insignificant spots of blue and green paint. (The spots have no effect on the scattering.) Suppose the differential cross section in the center-of-mass system for the detection of blue particles is $(d\sigma/d\Omega)(\theta,\phi)$. What is the differential cross section $(d\sigma/d\Omega)_{\rm dc}(\theta,\phi)$ for the detection of particles when we don't care about the color?

(b) Consider the scattering of two identical particles of spin j in quantum mechanics. Work in the center-of-mass system, and let $\mu = m/2$ be the reduced mass. Consider in particular three cases: $j = 0$, $j = \frac{1}{2}$, and $j = 1$. Organize the eigenstates of $H_0 = p^2/2\mu$ as tensor products of spatial states times spin states; make the spin states eigenstates of S^2 and S_z, where $\mathbf{S} = \mathbf{J}_1 + \mathbf{J}_2$, and make the spatial states properly symmetrized or antisymmetrized plane waves. Let the initial spin state be $|S_i M_{Si}\rangle$ and the final one be $|S_f M_{Sf}\rangle$. Since potential scattering cannot flip the spin, the cross section will be proportional to $\delta(S_i, S_f)\delta(M_{Si}, M_{Sf})$. Find the differential cross section in terms of

$$\tilde{U}_+ = \tilde{U}(\mathbf{k}_f + \mathbf{k}_i), \qquad \tilde{U}_- = \tilde{U}(\mathbf{k}_f - \mathbf{k}_i), \tag{1}$$

where $\tilde{U}$ is defined as in Eq. (29.59). Use the fact that $U(\mathbf{r}) = U(-\mathbf{r})$ to simplify the result as much as possible. Use notation like that in Eq. (29.61).

(c) For the three cases $j = 0$, $j = \frac{1}{2}$, $j = 1$, assume that the initial state is unpolarized and that we do not care about the final spin state. Find the differential cross section in terms of the quantities $a = |\tilde{U}_+|^2$, $b = |\tilde{U}_-|^2$, and $c = \mathrm{Re}(\tilde{U}_+^* \tilde{U}_-)$.

(d) Work out the answer for the case of Coulomb scattering of two electrons, and compare to the classical Rutherford formula, Eq. (29.74). Express your answer in a notation similar to that of Eq. (29.74). The cross term you get in applying the results of part (c) to Coulomb scattering is actually incorrect; the trouble is that plane waves do not adequately represent the unbound Coulomb wave functions, which have long range, logarithmic phase shifts. The correct answer is called the Mott cross section, which we will discuss later in class.

2. Short-range potentials give rise to s-wave scattering at sufficiently low energies, as discussed in class. The condition is $\lambda \gg R$, where λ is the de Broglie wavelength of the incident wave, and R is the range of the potential. In this case only the one term $\ell = 0$ contributes to the partial wave expansion of the scattering amplitude, and the amplitude itself is characterized by a single parameter, the $\ell = 0$ phase shift δ_0. Any other short range potential with the same value of δ_0 will behave the same insofar as low energy scattering is concerned. For this reason we often replace a real potential by a delta function, multiplied by some strength parameter g that we can adjust to make the phase shift δ_0 come out right, since this is mathematically simpler than the true potential. This partly explains the popularity of δ-function potentials in theoretical models.

In Bose-Einstein condensates (cold gases of bosonic atoms), the temperature and density are such that for atom-atom scattering $\lambda \gg R$, where R is the radius of the atom and where λ, the de Broglie wavelength, is comparable to the interparticle separation (this is required for the condensation). Thus replacing the atom-atom potential by a δ-function is a good approximation.

Typically Bose-Einstein condensates are gases in which the atoms are moving in some external, confining potential, call it $V(\mathbf{r})$. Based on what we have said, the Hamiltonian for a system of N identical bosonic atoms of spin 0 in the potential V is

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + V(\mathbf{r}_i) \right) + g \sum_{i<j} \delta^3(\mathbf{r}_i - \mathbf{r}_j). \tag{2}$$

This differs from the Hamiltonian we used in studying the electrons in atoms in three respects: (1) the external potential is V instead of Z/r; (2) the interaction potentials are δ-functions instead of Coulomb interactions; (3) the particls are spin-0 bosons.

Write down a trial wave function for the ground state of this system of bosons that is as close in spirit as possible to the Hartree-Fock trial wave function in atoms, given that these are bosons instead of fermions. Derive a self-consistent, three-dimensional Schrödinger-like equation that must be satisfied to minimize the energy of the system. This equation is the starting point of much current research into Bose-Einstein condensates.

3. The strange thing about scattering from a hard sphere in the limit $ka \gg 1$ is that the total cross section is $2\pi r^2$, not πr^2, the geometrical cross section. When the wave length is short, we expect quantum mechanics to agree with classical mechanics, but it does not in this case.

(a) Work out the classical differential cross section $d\sigma/d\Omega$ for a hard sphere of radius a, and integrate it to get the total cross section σ.

(b) In problem 6.1 (last semester), you worked out the far field wave function $\psi(x, y, z)$, for $z \gg ka^2$, when a plane wave e^{ikz} traveling in the positive z-direction strikes a screen in the x-y plane with a circular hole of radius a cut out. In that problem the hole was centered on the origin. The solution was worked out for $\theta = \rho/z \ll 1$ (the paraxial approximation), where $\rho = \sqrt{x^2 + y^2}$. You can refer to the posted solution of that problem if you wish.

By subtracting this solution from the incident wave e^{ikz}, you get the far field wave function when a plane wave e^{ikz} strikes the *complementary* screen, that is, just a disk of radius a at the origin.

It turns out this wave field is the same as the wave field in hard sphere (of radius a) scattering in the limit $ka \gg 1$, if measured in the forward direction. That is because for forward scattering from a hard scatterer, the physics is dominated by diffraction, so it is only the projection of the scatterer onto the x-y plane that matters.

Write the scattered wave as $(e^{ikr}/r)f(\theta)$, express r as a function of z and θ for small θ, expand out to lowest order in θ, and compare to the asymptotic wave field to get an expression for the scattering amplitude for small angles θ. According to the optical theorm, the total cross section is given in terms of the imaginary part of the forward differential cross section by

$$\sigma = \frac{4\pi}{k} \operatorname{Im} f(0). \tag{3}$$

Use this formula to compute σ.

(c) Show that $d\sigma/d\Omega$ in the forward direction has a narrow peak of width $\Delta\theta \sim 1/ka \ll 1$. Write down an integral giving the contribution of this forward peak to the total cross section in terms of the first root b of the Bessel function J_1. You can approximate $\sin\theta = \theta$ in this integral, since θ is small. It turns out that the value of this integral does not change much if the upper limit is extended to infinity. Use the integral

$$\int_0^\infty \frac{dx}{x}\, J_1(x)^2 = \frac{1}{2}, \tag{4}$$

to find the contribution of the forward peak to the total cross section. (See Gradshteyn and Ryzhik, integral number 6.538.2.)