Physics 221B
Spring 2008
Homework 15
Due Friday, January 25 at 5pm

Note: This homework covers the material from the last week of lecture of Physics 221A. If you were not in 221A last fall then you need not do this homework.

Reading Assignment: Sakurai, pp. 313–316; Notes 23; lecture notes on variational method.

1. First some background. The Coulomb potential gives to an infinite number of bound states, because it is a long-range potential. Other, short range, potentials have only a finite number of bound states. In a 3-dimensional problem, the number of bound states can range from zero to infinity, that is, if the potential is weak enough and short range enough, there may not be any bound states at all. In one dimension, however, a potential which is overall attractive, in a certain sense, always possesses at least one bound state, even if it is very weak.

Consider a 1-dimensional problem, with Hamiltonian

$$H = \frac{p^2}{2m} + V(x), \tag{1}$$

where $-\infty < x < +\infty$. Assume that $V(x)$ vanishes for $|x| > R$. For $|x| < R$, the potential is allowed to do almost anything (it may be positive in some places, and negative in others). Write down an expression for the expectation value of the energy for the Gaussian wave function,

$$\psi(x) = \frac{1}{\sqrt{a\sqrt{\pi}}} \exp\Bigl(-\frac{x^2}{2a^2}\Bigr), \tag{2}$$

and evaluate the integrals you can evaluate. Use the result to show that a bound state exists if the potential satisfies

$$\int V(x)\, dx < 0. \tag{3}$$

We may call such a potential "overall attractive."

Now try the same trick in 3 dimensions and show that it does not work. Assume $V(\mathbf{r}) = 0$ for $r > R$, and use the wave function

$$\psi(\mathbf{r}) = \frac{1}{\bigl(a\sqrt{\pi}\bigr)^{3/2}} \exp\Bigl(-\frac{r^2}{2a^2}\Bigr). \tag{4}$$

2. Consider a central force problem in three dimensions, with potential $V(r)$. Suppose that it has at least one bound state. Use the variational principle to prove that the ground state is an s-wave.

3. A problem on the hyperfine interaction in hydrogen.

(a) Equation (23.43) of the notes was derived in the case $\ell \neq 0$. Show that it also applies in the case $\ell = 0$. Hint: Use the fact that the components of the tensor T_{ij}, defined in Eq. (23.11), are r^2 times linear combinations of the $Y_{2m}(\theta, \phi)$, for $m = -2, \ldots, +2$. This is related to the fact that T_{ij} is the Cartesian version of an order 2 irreducible tensor.

(b) Our analysis of the hyperfine interaction in hydrogen has included the energy of interaction of the electron with the magnetic dipole field produced by the proton, but it seems that we have not included the energy of interaction of the proton spin with the magnetic field produced by the electron. As seen by the proton, the electron produces a magnetic field for two reasons: first, it is a charge in motion, therefore a current, which makes a magnetic field. This is the magnetic field due to the orbital motion of the electron. Next, the electron has a magnetic moment of its own, which makes a dipole magnetic field. This is the magnetic field produced by the spin of the electron.

Work out an expression for the energy of interaction of the proton spin with the magnetic field produced by the orbital motion of the electron. Follow the analysis of the spin-orbit interaction in Sec. 21.3, but run it backwards. That is, putting primes on the fields in the electron rest frame and no primes on fields in the proton rest frame, use Coulomb's law to write down the field $\mathbf{E}'$ of the electron in its own rest frame, then Lorentz transform to the lab frame to get $\mathbf{B}$ (call this $\mathbf{B}_{\text{orb}}$, the magnetic field due to the orbital motion of the electron). Then the energy of interaction of the proton with this magnetic field is $-\boldsymbol{\mu}_p \cdot \mathbf{B}_{\text{orb}}$, where $\boldsymbol{\mu}_p$ is the proton magnetic moment. Notice that unlike the analysis of Sec. 21.3, there is no factor of $\frac{1}{2}$ from Thomas precession, because the proton frame is not accelerated.

Now use Eq. (23.15) in the limit $R \to 0$ to obtain the magnetic field produced by the dipole moment of the electron at the position of the proton. Call this $\mathbf{B}_{\text{spin}}$, and write down an expression for the energy of interaction $-\boldsymbol{\mu}_p \cdot \mathbf{B}_{\text{spin}}$.

If you add these terms to the Hamiltonian (Eqs. (23.23) plus (23.24)), does it change the energy shifts (23.43)? These energy shifts are confirmed experimentally (for example, by the 21 cm line). What is wrong?

(c) Compute the hyperfine splitting of the ground state of positronium in wavelength units. Notice that in positronium, the fine structure and hyperfine structure are of the same order of magnitude.

Now some remarks. The interesting thing about this calculation is that the answer based on what you now know is actually wrong, because it omits a virtual process (a Feynman diagram) in which the positron and electron annihilate into a photon, which then materialize back into a positron and electron.

The analysis of this process requires quantum field theory. The Hamiltonians we usually use in atomic, molecular and solid state physics, expressed in terms of a finite number of particles and a finite number of degrees of freedom, are only valid up to a certain degree of accuracy, beyond which interactions with the infinite degrees of freedom in various fields (electromagnetic, electron-positron, strong interactions, ...) cannot be ignored. The first place where this occurs in hydrogen is with the Lamb shift. In positronium, it happens at the level of the fine structure (in positronium, the hyperfine structure is considered part of the fine structure).