Physics 221A
Fall 2007
Homework 14
Due Monday, December 10, 2007 at 5pm

Note: This assignment is due on the Monday after the usual due date. This is to give you some extra time, since some of the questions pertain to the lecture of Tuesday, December 4 (on the Zeeman effect).

Reading Assignment: Sakurai, pp. 302–311; Notes 20, 21, 22. See also the supplementary notes on Thomas precession (only if you're interested).

1. Use "nearly degenerate perturbation theory" (see Sec. 19.16 of the Notes) to analyze the effect of an external electric field on the $2s_{1/2}$ and $2p_{1/2}$ levels of hydrogen. This is not a model calculation like that done in class, where we looked at the Stark effect on the $n = 2$ levels of hydrogen in the electrostatic model; rather, we want to know what would happen in real hydrogen. Thus, you should use the fine structure model of hydrogen, including the spin, and take into account the Lamb shift, which suppresses the $2p_{1/2}$ level about 1.06 GHz below the $2s_{1/2}$ level. The $2p_{3/2}$ level is 10.9 GHz above the $2p_{1/2}$ level, and can be ignored for the purpose of this problem. We also ignore the hyperfine structure in this problem.

Show that the energy shifts are quadratic in the field strength for small amplitude electric fields, but that they become linear at larger strengths. Thus there is a threshold field strength, call it F_0, where the behavior changes from quadratic to linear. Estimate F_0 in Volts/cm. Sketch the energy levels as a function of the electric field strength F. In the limit $F \gg F_0$, you will obtain a linear relationship between ΔE and F, but it will not be the same linear relationship derived in class (also shown in the book's Eq. (5.2.20)]. Explain why. Explain how this problem provides an example of Kramer's degeneracy (and use this fact to cut your work in half).

In this problem you may use the fact mentioned in class,

$$\langle 200|z|210\rangle = -3a_0, \tag{1}$$

where the states indicated are the purely orbital states $|n\ell m\rangle$ [with wave functions $R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$]. (Here a_0 is the Bohr radius.)

2. The projection theorem (Sakurai, pp. 241–242) is useful in evaluating matrix elements for the Zeeman effect and for the hyperfine structure of hydrogen. The following is a different approach from Sakurai's.

Show that if $\mathbf{A}$ is a vector operator, then

$$[J^2, [J^2, \mathbf{A}]] = 2\hbar^2(\mathbf{A}J^2 + J^2\mathbf{A}) - 4\hbar^2(\mathbf{A}\cdot\mathbf{J})\mathbf{J}. \tag{2}$$

Then use this to show that

$$j(j+1)\hbar^2\langle\gamma' jm'|\mathbf{A}|\gamma jm\rangle = \langle\gamma' jm'|(\mathbf{A}\cdot\mathbf{J})\mathbf{J}|\gamma jm\rangle, \tag{3}$$

where the notation of Notes 11 is used. Finally, show how Sakurai's Eq. (3.10.40) follows from Eq. (3).

3. In this problem we aim to do a realistic calculation of the effect of magnetic fields on the $n = 2$ states of hydrogen. (For simplicity we will set $Z = 1$ throughout.) We will allow the magnetic field to take on any value, so that we will not assume that the Zeeman term is necessarily small or large in comparison to the fine structure terms. (We will, however, ignore the Lamb shift.) In class we studied various limiting cases. First, when B is so weak that the spin-orbit term dominates the Zeeman term, we found that the energy shifts were

$$\Delta E = g\mu_B B m_j, \tag{4}$$

where g is the Landé g-factor,

$$g = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}. \tag{5}$$

These are equivalent to Sakurai's Eq. (5.3.32). Next, when B was so strong that the spin-orbit term could be neglected, we found

$$\Delta E = \mu_B B(m_\ell + 2m_s). \tag{6}$$

See Sec. 22.4. The book does similar calculations, but it ignores the Darwin and relativistic kinetic energy corrections, which is not realistic because they are of the same order of magnitude as the spin-orbit term.

Consider therefore the Hamiltonian,

$$H = H_0 + H_1, \tag{7}$$

where

$$H_1 = H_{RKE} + H_{SO} + H_D + H_Z. \tag{8}$$

See Eqs. (21.4), (21.5) and (22.14) for the definitions of these various terms. Take the magnetic field to point along the z-axis, $\mathbf{B} = B\hat{\mathbf{z}}$. The term H_Z is proportional to the applied field B, which is a parameter of the perturbing Hamiltonian. You are to evaluate the corrections to the energy levels due to this perturbation, as a function of B, using first-order perturbation theory. To simplify the notation, I suggest that you use atomic units throughout. Also, I recommend that you introduce the dimensionless variable x to represent the strength of the magnetic field,

$$x = \frac{B}{B_1}, \tag{9}$$

where

$$B_1 = \frac{e^7 m^2}{\hbar^5 c} = \alpha B_0, \tag{10}$$

where B_0 is defined by Eq. (22.4). This definition is useful for the present problem, because it makes $x = 1$ approximately the value for which the Zeeman term is comparable to the fine structure term.

(a) Out of the list of operators, L^2, L_x, L_y, L_z, S^2, S_x, S_y, S_z, J^2, J_x, J_y, J_z, π, indicate which commute with H_0 and which commute with the entire Hamiltonian H. Use this information to choose a convenient basis in the 8-dimensional subspace of the $n = 2$ degenerate energy levels of H_0, for which the perturbing Hamiltonian will be as diagonal as possible.

(b) Let ΔE be the difference between the true energy levels of H (including the perturbation) and the 8-fold degenerate level $E_2 = -1/8 = -1/(2 \cdot 2^2)$ of H_0. Find all eight levels as a function of x in atomic units.

(c) Expand these results out for small x, and show that they agree with Eq. (4). Also expand the results for large x, and show that the results agree with Eq. (6).