Physics 221A
Fall 2007
Homework 11
Due Friday, November 16, 2007

Reading Assignment: Sakurai, pp. 203–217, 232–238; rest of Notes 15, pp. 1–24 of Notes 16.

I have posted a reprint from Jagdish Mehra, *The Beat of a Different Drum*, pp. 226–229, containing Feynman's recollections about his use of path integrals to obtain a covariant formulation of quantum electrodynamics. The section is titled, "I can do that for you!"

1. I won't ask you to work out any numerical values of Clebsch-Gordan coefficients, but you should be comfortable in doing so. The following is a related problem.

Consider the angular momentum problem $\ell \otimes \frac{1}{2}$, where ℓ is arbitrary and could be very large. We write $j = \ell + \frac{1}{2}$ or $j = \ell - \frac{1}{2}$ for the resulting angular momentum. By beginning with the doubly stretched state,

$$|\ell + \tfrac{1}{2}, \ell + \tfrac{1}{2}\rangle = |\ell\ell\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle, \tag{1}$$

apply lowering operators to construct the states $|\ell + \frac{1}{2}, m\rangle$ for m going down to $\ell - \frac{5}{2}$. By this time a pattern should be evident; guess it, and prove that it is right by induction, to obtain a general formula for $|\ell + \frac{1}{2}, m\rangle$. You may simplify notation by omitting the total angular momenta ℓ and $\frac{1}{2}$ in the kets on the right hand sides of your equations, because they are always the same; for example, the RHS of Eq. (1) can be written simply $|\ell\rangle|\frac{1}{2}\rangle$.

Now construct the stretched state for $j = \ell - \frac{1}{2}$, namely $|\ell - \frac{1}{2}, \ell - \frac{1}{2}\rangle$. Use the standard phase convention given by Eq. (15.31). Then lower this enough times to see a pattern, guess it, and prove it by induction. Note the Useful Formula section below.

2. The following problem will help you understand irreducible tensor operators better. Let $\mathcal{E}$ be a ket space for some system of interest, and let $\mathcal{A}$ be the space of linear operators that act on $\mathcal{E}$. For example, the ordinary Hamiltonian is contained in $\mathcal{A}$, as are the components of the angular momentum $\mathbf{J}$, the rotation operators $U(\mathsf{R})$, etc. The space $\mathcal{A}$ is a vector space in its own right, just like $\mathcal{E}$; operators can be added, multiplied by complex scalars, etc. Furthermore, we may be interested in certain subspaces of $\mathcal{A}$, such as the 3-dimensional space of operators spanned by the components V_x, V_y, V_z of a vector operator $\mathbf{V}$.

Now let $\mathcal{S}$ be the space of linear operators that act on $\mathcal{A}$. We call an element of $\mathcal{S}$ a "super" operator because it acts on ordinary operators; ordinary operators in $\mathcal{A}$ act on kets in $\mathcal{E}$. We will denote super-operators with a hat, to distinguish them from ordinary operators. (This terminology has nothing to do with supersymmetry.)

Given an ordinary operator $A \in \mathcal{A}$, it is possible to associate it in several different ways with a super-operator. For example, we can define a super operator $\hat{A}_L$, which acts by left multiplication:

$$\hat{A}_L X = AX, \tag{2}$$

where X is an arbitrary ordinary operator. This equation obviously defines a linear super-operator, i.e., $\hat{A}_L(X+Y) = \hat{A}_L X + \hat{A}_L Y$, etc. Similarly, we can define a super-operator associated with A by means of right multiplication, or by means of the forming of the commutator, as follows:

$$\begin{aligned} \hat{A}_R X &= XA, \\ \hat{A}_C X &= [A, X]. \end{aligned} \tag{3}$$

There are still other ways of associating an ordinary operator with a super-operator. Let R be a classical rotation, and let $U(\mathsf{R})$ be a representation of the rotations acting on the ket space $\mathcal{E}$. Thus, the operators $U(\mathsf{R})$ belong to the space $\mathcal{A}$. Now associate such a rotation operator $U(\mathsf{R})$ in $\mathcal{A}$ with a super-operator $\hat{U}(\mathsf{R})$ in $\mathcal{S}$, defined by

$$\hat{U}(\mathsf{R})X = U(\mathsf{R})\, X\, U(\mathsf{R})^\dagger. \tag{4}$$

Again, $\hat{U}(\mathsf{R})$ is obviously a linear super-operator.

(a) Show that $\hat{U}(\mathsf{R})$ forms a representation of the rotations, that is, that

$$\hat{U}(\mathsf{R}_1)\hat{U}(\mathsf{R}_2) = \hat{U}(\mathsf{R}_1\mathsf{R}_2). \tag{5}$$

This is easy.

Now let $U(\mathsf{R})$ be infinitesimal as in Eq. (16.15), and let

$$\hat{U}(\mathsf{R}) = 1 - \frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\hat{\mathbf{J}}. \tag{6}$$

(Here the hat on $\hat{\mathbf{n}}$ denotes a unit vector, while that on $\hat{\mathbf{J}}$ denotes a super-operator.) Express the super-operator $\hat{\mathbf{J}}$ in terms of ordinary operators. Write Eqs. (16.76) in super-operator notation. Work out the commutation relations of the super-operators $\hat{\mathbf{J}}$.

(b) Now write out nine equations, specifying the action of the three super-operators $\hat{J}_i$ on the the basis operators V_j. Write the answers as linear combinations of the V_j's. Then write out six more equations, specifying the action of the super raising and lowering operators, $\hat{J}_\pm$, on the three V_j.

Now find the operator A that is annihilated by $\hat{J}_+$. Do this by writing out the unknown operator as a linear combination of the V_j's, in the form

$$A = a_x V_x + a_y V_y + a_z V_z, \tag{7}$$

and then solving for the coefficients a_i. Show that this operator is an eigenoperator of $\hat{J}_z$ with eigenvalue $+\hbar$. In view of these facts, the operator A must be a "stretched" operator for $k = 1$; henceforth write T^1_1 for it. This operator will have an arbitrary, complex multiplicative constant, call it c. Now apply $\hat{J}_-$, and generate T^1_0 and T^1_{-1}. Choose the constant c to make T^1_0 look as simple as possible. Then write

$$T^1_q = \hat{\mathbf{e}}_q \cdot \mathbf{V}, \tag{8}$$

and thereby "discover" the spherical basis.

3. This problem concerns quadrupole moments and spins. It provides some background for the following problem.

(a) In the case of a nucleus, the spin Hilbert space $\mathcal{E}_{\text{spin}} = \text{span}\{|sm\rangle, m = -s, \ldots, +s\}$ is actually the ground state of the nucleus. It is customary to denote the angular momentum j of the ground state by s. This state is $(2s+1)$-fold degenerate. The nuclear spin operator $\mathbf{S}$ is really the restriction of the total angular momentum of the nucleus $\mathbf{J}$ to this subspace of the (much larger) nuclear Hilbert space.

Let A^k_q and B^k_q be two irreducible tensor operators on $\mathcal{E}_{\text{spin}}$. As explained in the notes, when we say "irreducible tensor operator" we are really talking about the collection of $2k+1$ operators obtained by setting $q = -k, \ldots, +k$. Use the Wigner-Eckart theorem to explain why any two such operators of the same order k are proportional to one another. This need not be a long answer.

Thus, all scalars are proportional to a standard scalar (1 is convenient), and all vector operators (for example, the magnetic moment $\boldsymbol{\mu}$) are proportional to a standard vector ($\mathbf{S}$ is convenient), etc.

For a given s, what is the maximum value of k? What is the maximum order of an irreducible tensor operator that can exist on space $\mathcal{E}_{\text{spin}}$ for a proton (nucleus of ordinary hydrogen)? A deuteron (heavy hydrogen)? An alpha particle (nucleus of helium)?

(b) Let $\mathbf{A}$ and $\mathbf{B}$ be two vector operators (on any Hilbert space, not necessarily $\mathcal{E}_{\text{spin}}$), with spherical components A_q, B_q, as in Eq. (16.64). As explained in the notes, A_q and B_q are

$k = 1$ irreducible tensor operators. As explained in Sec. 16.17, it is possible to construct irreducible tensor operators T^k_q for $k = 0, 1, 2$ out of the nine operators, $\{A_q B_{q'}, q, q' = -1, 0, 1\}$. Write out the nine operators $T^0_0, T^1_1, T^1_0, T^1_{-1}, T^2_2, \ldots, T^2_{-2}$ in terms of the Cartesian products $A_i B_j$. Just look up the Clebsch-Gordan coefficients. Show that T^0_0 is proportional to $\mathbf{A}\cdot\mathbf{B}$, that T^1_q is proportional to the spherical components of $\mathbf{A}\times\mathbf{B}$, and that T^2_q specifies the components of the symmetric and traceless part of the Cartesian tensor $A_i B_j$, which is

$$\frac{1}{2}(A_i B_j + A_j B_i) - \frac{1}{3}(\mathbf{A}\cdot\mathbf{B})\delta_{ij}.$$

(c) In classical electrostatics, the quadrupole moment tensor Q_{ij} of a charge distribution $\rho(\mathbf{r})$ is defined by

$$Q_{ij} = \int d^3\mathbf{r}\, \rho(\mathbf{r})[3r_i r_j - r^2\,\delta_{ij}], \tag{9}$$

where $\mathbf{r}$ is measured relative to some origin inside the charge distribution. The quadrupole moment tensor is a symmetric, traceless tensor. The quadrupole energy of interaction of the charge distribution with an external electric field $\mathbf{E} = -\nabla\phi$ is

$$E_{\text{quad}} = \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial^2\phi(0)}{\partial r_i \partial r_j}. \tag{10}$$

This energy must be added to the monopole and dipole energies, plus the higher multipole energies.

In the case of a nucleus, we choose the origin to be the center of mass, whereupon the dipole moment and energy vanish. The monopole energy is just the usual Coulomb energy $q\phi(0)$, where q is the total charge of the nucleus. Thus, the quadrupole term is the first nonvanishing correction. However, the energy must be understood in the quantum mechanical sense.

Let $\{\mathbf{r}_\alpha, \alpha = 1, \ldots, Z\}$ be the position operators for the protons in a nucleus. The neutrons are neutral, and do not contribute to the electrostatic energy. The electric quadrupole moment operator for the nucleus is defined by

$$Q_{ij} = e\sum_\alpha (3r_{\alpha i} r_{\alpha j} - r_\alpha^2\,\delta_{ij}), \tag{11}$$

where e is the charge of a single proton. In an external electric field, the nuclear Hamiltonian contains a term H_{quad}, exactly in the form of Eq. (10), but now interpreted as an operator.

The operator Q_{ij}, being symmetric and traceless, constitutes the Cartesian specification of a $k = 2$ irreducible tensor operator, that you could turn into standard form $T^2_q, q =$

$-2, \ldots, +2$ using the method of part (b) if you wanted to. We'll say with the Cartesian form here, however. When the operator Q_{ij} is restricted to the ground state (really a manifold of $2s+1$ degenerate states), it remains a $k=2$ irreducible tensor operator. According to part (a), it must be proportional to some standard irreducible tensor operator, for which $3S_iS_j - S^2\delta_{ij}$ is convenient. That is, we must be able to write

$$Q_{ij} = a(3S_iS_j - S^2\delta_{ij}), \tag{12}$$

for some constant a.

It is customary in nuclear physics to denote the "quadrupole moment" of the nucleus by the real number Q, defined by

$$Q = \langle ss|Q_{33}|ss\rangle, \tag{13}$$

where $|ss\rangle$ is the stretched state. Don't confuse Q_{ij}, a tensor of operators, with Q, a single number. Express a in terms of Q, and derive the form H_{int} in Sakurai's problem 29, p. 247. There is an error in Sakurai's formula; correct it.

4. Sakurai, problem 3.29, p. 247.

The existence of a quadruopole moment of the deuteron (a spin 1 particle) was discovered by Kellogg in 1939. This caused quite a stir at the time, because it indicates that the wave function for the relative motion of the proton and neutron is not spherically symmetric (it is not purely an s-wave). This means that the force between the proton and neutron is not describable purely in terms of a central force potential; instead, there is a significant spin-dependent contribution. Nuclear electric quadrupole moments are an important means of probing nuclear structure. Sakurai's problem seems to imply that the electric field specified by ϕ is an external field, but in practice it is the electric field produced by the cloud of electrons in an atom or molecule. Therefore the nuclear quadrupole moment gives rise to *hyperfine* splittings and shiftings of atomic energy levels. In general, magnetic hyperfine interactions (spin-spin interactions between atomic electrons and the nucleus) are of the same order of magnitude, and must be included in any realistic analysis of electric quadrupole interactions.

5. Consider the helium atom in a lab frame. The positions of the nucleus, electron 1 and electron 2 are $\mathbf{r}_n$, $\mathbf{r}_{e1}$ and $\mathbf{r}_{e2}$ respectively. The mass of the nucleus is M and that of the electron is m. Thus the laboratory Hamiltonian is

$$H = \frac{|\mathbf{p}_n|^2}{2M} + \frac{|\mathbf{p}_{e1}|^2}{2m} + \frac{|\mathbf{p}_{e2}|^2}{2m} + V(\mathbf{r}_n, \mathbf{r}_{e1}, \mathbf{r}_{e2}). \tag{14}$$

We will only be interested in the kinetic energy in this problem.

Let $\mathbf{R}$ be the center of mass position and let

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_{e1} - \mathbf{r}_n, \\ \mathbf{r}_2 &= \mathbf{r}_{e2} - \mathbf{r}_n, \end{aligned} \tag{15}$$

so that $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two electrons relative to the nucleus. Also define new momentum operators,

$$\mathbf{P} = -i\hbar \frac{\partial}{\partial \mathbf{R}}, \quad \mathbf{p}_1 = -i\hbar \frac{\partial}{\partial \mathbf{r}_1}, \quad \mathbf{p}_2 = -i\hbar \frac{\partial}{\partial \mathbf{r}_2}. \tag{16}$$

Express the kinetic energy as a function of these new momentum operators. You will obtain a term proportional to $\mathbf{p}_1 \cdot \mathbf{p}_2$. This is called a “mass polarization” term.

The usual simple minded treatment of helium treats the nucleus as infinitely heavy. In this approach there are no mass polarization terms. Explain why its a good approximation to neglect those terms (thus, the usual approach is ok).

Useful Formulas

The following are some useful formulas, which can be derived by the methods of problem 1. We combine $\ell \otimes 1$, and find, for the three cases $j = \ell + 1$, $j = \ell$, and $j = \ell - 1$, the following:

$$\begin{aligned} |\ell+1, m\rangle &= \sqrt{\frac{(\ell+m+1)(\ell+m)}{(2\ell+2)(2\ell+1)}}\,|m-1\rangle|1\rangle \\ &+ \sqrt{\frac{(\ell-m+1)(\ell+m+1)}{(\ell+1)(2\ell+1)}}\,|m\rangle|0\rangle + \sqrt{\frac{(\ell-m)(\ell-m+1)}{(2\ell+2)(2\ell+1)}}\,|m+1\rangle|-1\rangle. \end{aligned} \tag{17a}$$

$$\begin{aligned} |\ell m\rangle &= -\sqrt{\frac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}}\,|m-1\rangle|1\rangle \\ &+ \frac{m}{\sqrt{\ell(\ell+1)}}|m\rangle|0\rangle + \sqrt{\frac{(\ell-m)(\ell+m+1)}{2\ell(\ell+1)}}\,|m+1\rangle|-1\rangle. \end{aligned} \tag{17b}$$

$$\begin{aligned} |\ell-1, m\rangle &= \sqrt{\frac{(\ell-m)(\ell-m+1)}{2\ell(2\ell+1)}}\,|m-1\rangle|1\rangle \\ &- \sqrt{\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}}\,|m\rangle|0\rangle + \sqrt{\frac{(\ell+m+1)(\ell+m)}{2\ell(2\ell+1)}}\,|m+1\rangle|-1\rangle. \end{aligned} \tag{17c}$$