Physics 221A
Fall 2007
Homework 10
Due Friday, November 9, 2007

Reading Assignment: Rest of Notes 13, Notes 14, beginning of Notes 15, lecture notes for the week.

I have also posted some notes on converting between SI and Gaussian units. They are somewhat incomplete but may be helpful.

1. Consider wave functions $f(\theta,\phi)$ on the unit sphere, as in Sec. 13.5. Using the differential operators in Sec. 13.4, find the simultaneous eigenfunction of L^2 and L_z with eigenvalues $2\hbar^2$ and $\hbar$, respectively (that is, the case $\ell = 1$). Normalize this wave function but leave leave a phase factor that will be determined later. Now apply L_- twice to this state fill out the standard basis vectors in an irreducible subspace. The wave functions are proportional to Y_{1m} for $m = 1, 0, -1$. By requiring Y_{10} to be real and positive at the north pole, determine the phase. Compare your answers to a table of $Y_{\ell m}$'s.

2. Some problems concerning orbital angular momentum in the momentum representation.

(a) Consider a spinless particle moving in three-dimensional space. It was shown in class that the standard angular momentum basis consists of wavefunctions of the form $u_n(r)Y_{\ell m}(\theta,\phi)$, where $u_n(r)$ is an arbitrary basis of radial wave functions. The wave functions being referred to here are configuration space wave functions, $\psi(\mathbf{r})$ or $\psi(r,\theta,\phi)$. Consider the wave functions $\phi(\mathbf{p})$ in the momentum representation. Let (p,β,α) be spherical coordinates in momentum space, that is,

$$\begin{aligned} p_x &= p\sin\beta\cos\alpha, \\ p_y &= p\sin\beta\sin\alpha, \\ p_z &= p\cos\beta. \end{aligned} \tag{1}$$

Find the form of the wavefunctions that make up the standard angular momentum basis in momentum space.

(b) Let $\psi(\mathbf{r})$ be a wave function in three dimensions, and let $\psi'(\mathbf{r})$ be the rotated wave function corresponding to rotation matrix $\mathsf{R} \in SO(3)$. According to Eq. (13.13),

$$\psi'(\mathbf{r}) = \psi(\mathsf{R}^{-1}\mathbf{r}). \tag{2}$$

Use this fact and the usual expression for the momentum space wave function,

$$\phi(\mathbf{p}) = \int \frac{d^3\mathbf{r}}{(2\pi\hbar)^{3/2}}\, e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\, \psi(\mathbf{r}), \tag{3}$$

to find a relation between $\phi'(\mathbf{p})$ and $\phi(\mathbf{p})$.

(c) A useful formula in scattering theory is the expansion of a plane wave in terms of free particle solutions of definite angular momentum. It is

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\ell=0}^{\infty} e^{i\ell\pi/2}\,(2\ell+1) j_\ell(kr) P_\ell(\cos\gamma), \tag{4}$$

where γ is the angle between $\mathbf{r}$ and $\mathbf{k}$, where j_ℓ is a spherical Bessel function, and where P_ℓ is a Legendre polynomial.

Suppose $\psi(\mathbf{r}) = u(r)Y_{\ell m}(\theta,\phi)$. Show that $\phi(\mathbf{p}) = v(p)Y_{\ell m}(\beta,\alpha)$, and find a one-dimensional integral transform connecting the radial functions $u(r)$ and $v(p)$.

3. This problem involves some exercise with expectation values of powers of r in hydrogen-like atoms. These expectation values are useful in perturbation theory and other places. There are various ways to evaluate these expectation values; for example, one can use the generating function of the Laguerre polynomials. But I think the following method is somewhat easier, once you get it going.

(a) Write out the radial Schrödinger equation for a hydrogen-like atom. ("Hydrogen-like" means $V(r) = -Ze^2/r$.) Show that if $a = \hbar^2/\mu Z e^2$, then

$$\frac{d^2u}{dr^2} + \Big[-\frac{\ell(\ell+1)}{r^2} + \frac{2}{ar} - \frac{1}{n^2a^2}\Big]u = 0, \tag{5}$$

where u is the normalized radial wave function corresponding to quantum numbers n and ℓ. Now multiply this by $r^{k+1}(du/dr)$ and integrate from 0 to ∞. Use integration by parts to show that

$$(k+1)\int_0^\infty r^k\Big(\frac{du}{dr}\Big)^2 dr = \frac{k+1}{n^2a^2}\langle r^k\rangle - \frac{2k}{a}\langle r^{k-1}\rangle + \ell(\ell+1)(k-1)\langle r^{k-2}\rangle. \tag{6}$$

In the integrations, you may assume that $k > -2\ell-2$, which will cause the boundary terms to vanish.

(b) Now multiply (5) by $r^k u$ and do more integration by parts, and combine the result with (6) to show that

$$\frac{k+1}{n^2}\langle r^k\rangle - (2k+1)a\langle r^{k-1}\rangle + \frac{a^2k}{4}\big[(2\ell+1)^2 - k^2\big]\langle r^{k-2}\rangle = 0. \tag{7}$$

This is called the *Kramer's relation.*

(c) Use (7) to find $\langle r^k \rangle$ for $k = -1$, $k = 1$, and $k = 2$. Notice that you cannot evaluate $\langle 1/r^2 \rangle$ by this method. For that you need to face up to generating functions, or some other method. However, given that

$$\left\langle \frac{1}{r^2} \right\rangle = \frac{1}{a^2 n^3 (\ell + \frac{1}{2})}, \tag{8}$$

find $\langle 1/r^3 \rangle$. The latter is needed for the fine structure perturbations and the Zeeman effect.

4. About wave functions of particles, including spin.

(a) The wave function of a particle with spin was defined in Eq. (15.12). Also, it was explained in class that if rotations act on ket space $\mathcal{E}_1$ by means of operators $U_1(\mathsf{R})$, and on ket space $\mathcal{E}_2$ by means of operators $U_2(\mathsf{R})$, then they act on $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ by means of operators $U(\mathsf{R}) = U_1(\mathsf{R}) \otimes U_2(\mathsf{R})$. The latter product would often be written in physics literature without the $\otimes$, that is, simply $U(\mathsf{R}) = U_1(\mathsf{R})U_2(\mathsf{R})$. Here we are thinking of combining orbital and spin ket spaces.

If $\psi_m(\mathbf{r})$ is the wave function of a spinning particle in state $|\psi\rangle$, then what is the wave function of the particle in the rotated state, $U(\mathsf{R})|\psi\rangle$?

(b) The *Pauli equation* is the Schrödinger equation for an electron interacting with electric and magnetic fields, including the spin degrees of freedom. The Pauli Hamiltonian is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\Phi - \boldsymbol{\mu} \cdot \mathbf{B}, \tag{9}$$

and the wave function is understood to be that of a spin-$\frac{1}{2}$ particle with $q = -e$ and $g =$ the electron g-factor. Here Φ and $\mathbf{A}$ are the electromagnetic scalar and vector potentials.

Consider an electron in a central force potential $V(r)$, plus a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{b}}$. Let $\omega_0 = eB/mc$. Use the gauge $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$. Consider the time-dependent Pauli equation for the electron,

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle, \tag{10}$$

where H is the Pauli Hamiltonian (9) and where $q\Phi = V(r)$.

Define a new state $|\phi(t)\rangle$ by

$$|\psi(t)\rangle = U(\hat{\mathbf{b}}, \omega t)|\phi(t)\rangle, \tag{2}$$

where $U(\hat{\mathbf{b}}, \omega t)$ is a rotation operator that rotates the whole system (orbital and spin degrees of freedom). This means that $|\phi(t)\rangle$ is the state in a frame rotating with angular velocity ω about the axis $\hat{\mathbf{b}}$.

Find a frequency ω that eliminates the effect of the magnetic field on the orbital motion of the particle, apart from the centrifugal potential which is proportional to $(\mathbf{b}\times\mathbf{r})^2$. Find a frequency ω that eliminates the effect of the magnetic field on the spin. Express your answers as some multiple of ω_0. Can you eliminate the effects of the magnetic field entirely, apart from the centrifugal potential?

5. In lecture we worked out the energy levels (the Landau levels) and eigenfunctions for a "spinless" electron in a uniform magnetic field. See the lecture notes for Sept. 27. I mentioned that this was not a good approximation, because if you include the spin it does more than just make small perturbations in the levels you found without spin.

Including the spin, find the energy levels of an electron in a uniform magnetic field. Express your answer in terms of $\omega_0 = eB/mc$, the orbital frequency of a classical electron in a uniform magnetic field. This is a short problem, and does not require any lengthy calculations.