Physics 221A
Fall 2007
Homework 1
Due Thursday, September 6, 2007

Reading Assignment: Sakurai, pp. 1–23, Notes 1, Notes 2, pp. 1–6.

Note: This homework is due at 5pm on Thursday, September 6. Homework is to be turned in to the 221A box in the reading room, 251 LeConte. See the web site for homework policy.

1. Consider the 2×2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) Prove that

$$\exp(-i\theta\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) = I\cos\theta - i(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})\sin\theta, \tag{2}$$

where

$$\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{x}} + \sigma_y\hat{\mathbf{y}} + \sigma_z\hat{\mathbf{z}}. \tag{3}$$

Here $\hat{\mathbf{n}}$ is an arbitrary unit vector, and θ an arbitrary angle.

(b) Prove that, given any two vector operators $\mathbf{A}$, $\mathbf{B}$ that commute with $\boldsymbol{\sigma}$ (but not necessarily with each other), we have the identity,

$$(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} \times \mathbf{B}). \tag{4}$$

Two vector operators are considered to commute if all of their components commute. Note that in general, $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$, and $\mathbf{A} \times \mathbf{B} \neq -\mathbf{B} \times \mathbf{A}$.

2. Lie algebraic techniques for operators.

(a) Consider two operators A, B that do not necessarily commute. Show that

$$e^A Be^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots. \tag{5}$$

Hint: Replace A by λA, where λ is a parameter, and let the left-hand side be $F(\lambda)$. Find a differential equation satisfied by $F(\lambda)$, and solve it.

(b) Let $A(\lambda)$ be an operator that depends on a continuous parameter λ. Derive the following operator identity:

$$e^A \frac{d(e^{-A})}{d\lambda} = -\sum_{n=0}^{\infty} \frac{1}{(n+1)!} L_A^n \left(\frac{dA}{d\lambda}\right), \tag{6}$$

where

$$L_A(X) = [A, X], \quad L_A^2(X) = [A, [A, X]], \quad \ldots, \tag{7}$$

where X is an arbitrary operator. Do not assume that A commutes with $dA/d\lambda$.

In the following problems, you may assume wherever necessary that you are dealing with a finite-dimensional Hilbert space.

3. Some easy proofs from Notes 1.

(a) Show that Eq. (1.48) follows from Eq. (1.47).

(b) Prove Eqs. (1.55), (1.58), and (1.59).

(c) Prove that the product of two Hermitian operators is Hermitian if and only if they commute.

4. In general, there is little interest in the eigenkets and eigenbras of an arbitrary operator. Hermitian operators are an exception; so are anti-Hermitian and unitary operators. We define an operator to be *normal* if it commutes with its Hermitian conjugate, $[A, A^\dagger] = 0$. Notice that Hermitian, anti-Hermitian, and unitary operators are normal.

(a) Show that if A is normal, and $A|u\rangle = a|u\rangle$ for some nonzero $|u\rangle$, then $A^\dagger|u\rangle = a^*|u\rangle$. Thus, the eigenbras of A are the Hermitian conjugates of the eigenkets, and the left spectrum is identical to the right spectrum. Hint: it is not necessary to introduce orthonormal bases or anything of the kind.

(b) Show that the eigenspaces corresponding to distinct eigenvalues of a normal operator are orthogonal. This is a generalization of the easy and familiar proof for Hermitian operators.

5. Some things that will be needed in subsequent problems.

(a) If A is an observable, show that

$$P_n = \prod_{k \neq n} \frac{A - a_k}{a_n - a_k}, \tag{8}$$

where P_n is the projector onto the n-th eigenspace of A. This shows that the projector P_n is a function of the operator A (see Sec. 1.20).

(b) Show that if

$$\langle \psi | A | \psi \rangle = 0 \tag{9}$$

for all $|\psi\rangle$, then $A = 0$. (A is not necessarily Hermitian.) Would this be true on a real vector space? (A real vector space is one in which the coefficients allowed in forming linear combinations of basis vectors are real. In a complex vector space, such as Hilbert space, these coefficients are allowed to be complex.)

(c) Let U be a linear operator, and let $|\psi'\rangle = U|\psi\rangle$. Show that $\langle \psi' | \psi' \rangle = \langle \psi | \psi \rangle$ for all kets $|\psi\rangle$ if and only if U is unitary.