Physics 221A
Fall 1996
Notes 11
Representations of the Angular Momentum Operators and Rotations

In Notes 10, the angular momentum $\mathbf{J}$ of a quantum system was defined by Eq. (10.7) as the infinitesimal generator of rotations, and it was shown that the components of $\mathbf{J}$ must satisfy the commutation relations (10.16) in order that the quantum rotation operators should provide a representation of the classical rotations. We repeat these commutation relations here:

$$\boxed{[J_i, J_j] = i\hbar\, \epsilon_{ijk}\, J_k.} \tag{11.1}$$

Our strategy in finding a representation of the rotations is first to find a representation of the angular momentum commutation relations, that is, a set of three Hermitian operators (J_1, J_2, J_3) which satisfy Eq. (11.1), and then to exponentiate linear combinations of them according to

$$\boxed{U(\hat{\mathbf{n}}, \theta) = \exp\Big[-\frac{i}{\hbar}\theta(\hat{\mathbf{n}}\cdot\mathbf{J})\Big],} \tag{11.2}$$

which gives the rotation operators $U(\hat{\mathbf{n}}, \theta)$ themselves. In Notes 10, we carried out this strategy in detail, working with the specific representation of the angular momentum commutation relations given by $\mathbf{J} = (\hbar/2)\boldsymbol{\sigma}$, which gave us rotation operators for spin-$\frac{1}{2}$ systems.

In these notes we will solve the representation problem for the angular momentum commutation relations in all generality. That is, we will seek and find the most general vector (J_1, J_2, J_3) of Hermitian operators which satisfy Eq. (11.1). After we have done this, we will explore the properties of the rotation operators which are generated from the angular momentum operators by Eq. (11.2). We will find that the rotation operators created in this manner sometimes form a double-valued representation of the classical rotations, just as we found in the case of spin-$\frac{1}{2}$ rotations in Notes 10. Nevertheless, the existence of double-valued representations of the rotations does not diminish the importance of finding the representations of the angular momentum commutation relations.

Therefore let us assume that we have some vector space upon which three Hermitian operators (J_1, J_2, J_3) act, such that the commutation relations (11.1) are satisfied. We make no other assumptions about these operators or the vector space upon which they act, and, in particular, we make no assumptions about the dimensionality of the vector space. It pays to treat this problem in some generality, because there is a wide variety of circumstances in physical problems where such operators and commutation relations arise. For example, the vector space could be a ket space, in which case the operators (J_1, J_2, J_3) are ordinary operators in quantum mechanics. The ket space could belong to the spatial degrees of freedom for a quantum system (i.e., it could be a wave function space); it could be a ket space for spin degrees of freedom; it could be the tensor product of such spaces, perhaps representing a multiparticle system; or it could be a subspace of such spaces.

In fact, the vector space need not be a ket space. It could be a vector space of operators, as will be discussed in later notes on irreducible tensor operators and the Wigner-Eckart theorem. It could be ordinary 3-dimensional space, in which case we could "rediscover" the theory of classical rotations as in Notes 9. It could be a space of wave fields for a classical wave system, as in the multipole expansion for classical electromagnetic fields. There are many possibilities. Nevertheless, to be specific, in the following discussion we will proceed as if the vector space is a ket space, and we will use bra-ket notation for the vectors of the space.

We will seek the most general form which the vector of operators (J_1, J_2, J_3) can take, given that it satisfies Eq. (11.1). Among other things, we will be interested in the matrices which represent these operators in some appropriately chosen basis.

We begin by constructing the positive definite operator J^2,

$$J^2 = J_1^2 + J_2^2 + J_3^2, \tag{11.3}$$

which, as an easy calculation shows, commutes with all three components of $\mathbf{J}$:

$$[J^2, \mathbf{J}] = 0. \tag{11.4}$$

Since J^2 commutes with $\mathbf{J}$, it commutes also with any function of $\mathbf{J}$. An operator with this property is called a *Casimir operator.*

Since J^2 and $\mathbf{J}$ commute, we can construct simultaneous eigenkets of J^2 and one of the components of $\mathbf{J}$, which is conventionally taken to be J_3. We denote the eigenvalues of J^2 and J_3 by $\hbar^2 a$ and $\hbar m$, respectively, but at this point we make no assumptions about the allowed values which a and m might take on (positive, negative, integer, fraction, etc.). We denote the simultaneous eigenkets by $|\alpha a m\rangle$, where the index α is introduced to resolve possible degeneracies.

Actually, for pedagogical reasons, it is convenient to assume at first that there are no degeneracies, that is, that whatever simultaneous eigenkets of J^2 and J_3 might exist, they are always nondegenerate. When we have analyzed this case, it will be easy to come back and consider what happens when there are degeneracies. Therefore for now we will dispense with the index α, and write

$$\begin{aligned} J^2|am\rangle &= \hbar^2 a|am\rangle, \\ J_3|am\rangle &= \hbar m|am\rangle. \end{aligned} \tag{11.5}$$

Our first conclusion is that $a \geq 0$, which follows from the fact that J^2 is nonnegative definite. Therefore we can set

$$a = j(j+1), \tag{11.6}$$

where $j \geq 0$. This substitution simplifies some of the algebra to come later.

Next we define the ladder or raising and lowering operators,

$$J_\pm = J_1 \pm iJ_2, \tag{11.7}$$

which satisfy the commutation relations,

$$[J_3, J_\pm] = \pm\hbar J_\pm, \tag{11.8}$$

$$[J_+, J_-] = 2\hbar J_3, \tag{11.9}$$

$$[J^2, J_\pm] = 0. \tag{11.10}$$

We also have the relations,

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2, \tag{11.11}$$

$$J_-J_+ = J^2 - J_3(J_3 + \hbar), \tag{11.12}$$

$$J_+J_- = J^2 - J_3(J_3 - \hbar). \tag{11.13}$$

Now let us suppose that some nonzero eigenket $|jm\rangle$ with definite values j, m exists. Then by Eqs. (11.12) and (11.13), we have

$$\begin{aligned} \langle jm|J_-J_+|jm\rangle &= \hbar^2[j(j+1) - m(m+1)] \\ &= \hbar^2(j-m)(j+m+1) \geq 0, \end{aligned} \tag{11.14}$$

$$\begin{aligned} \langle jm|J_+J_-|jm\rangle &= \hbar^2[j(j+1) - m(m-1)] \\ &= \hbar^2(j+m)(j-m+1) \geq 0, \end{aligned} \tag{11.15}$$

where the inequality follows from the fact that the left hand sides are the squares of ket vectors. Consider first Eq. (11.14), which implies either

$$j - m \geq 0 \qquad \text{and} \qquad j + m + 1 \geq 0, \tag{11.16}$$

or

$$j - m \leq 0 \qquad \text{and} \qquad j + m + 1 \leq 0. \tag{11.17}$$

But Eq. (11.17) implies $j \leq -\frac{1}{2}$, which is impossible since $j \geq 0$, so Eq. (11.16) must be true. But Eq. (11.16) is equivalent to

$$-j - 1 \leq m \leq j. \tag{11.18}$$

Similarly, Eq. (11.15) implies,

$$-j \leq m \leq j + 1. \tag{11.19}$$

Taken together, Eqs. (11.18) and (11.19) imply

$$-j \leq m \leq +j, \tag{11.20}$$

which is a restriction on the eigenvalues j, m which are allowed.

Now we return to Eq. (11.14), and consider the case that the matrix element should vanish. This occurs only if $J_+|jm\rangle = 0$, which implies either $m = j$ or $m = -j - 1$. But in view of Eq. (11.20), the latter is impossible, so we see that $J_+|jm\rangle = 0$ if and only if $m = j$. Similarly, the vanishing of the matrix element in Eq. (11.15) implies $J_-|jm\rangle = 0$, which occurs if and only if $m = -j$. For all other values of m, $J_+|jm\rangle$ or $J_-|jm\rangle$ are nonzero kets (assuming as we are that $|jm\rangle$ is nonzero).

But it is easy to show that if $m \neq j$, then $J_+|jm\rangle$ is a simultaneous eigenket of J^2 and J_3 with eigenvalues $j(j+1)\hbar^2$ and $(m+1)\hbar$, respectively, for we have

$$\begin{aligned} J^2\big(J_+|jm\rangle\big) &= J_+ J^2|jm\rangle = j(j+1)\hbar^2\big(J_+|jm\rangle\big), \\ J_3\big(J_+|jm\rangle\big) &= (J_+J_3 + \hbar J_+)|jm\rangle = (m+1)\hbar\big(J_+|jm\rangle\big). \end{aligned} \tag{11.21}$$

From this it immediately follows that

$$m = j - n_1, \tag{11.22}$$

where n_1 is an integer, for if this were not so, we could successively apply J_+ to $|jm\rangle$ (which we are assuming to be nonzero), and generate nonzero kets with successively higher values of m until the rule (11.20) was violated. Similarly, we show that if $m \neq -j$, then $J_-|jm\rangle$ is a simultaneous eigenket of J^2 and J_3 with eigenvalues $j(j+1)\hbar^2$ and $(m-1)\hbar$, respectively, and from this we conclude that

$$m = -j + n_2, \tag{11.23}$$

where n_2 is another integer.

But by combining Eqs. (11.22) and (11.23), we conclude that $2j$ is an integer, so that the only allowed values of j are integral and half-integral,

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots. \tag{11.24}$$

We emphasize that the j values listed are those which are compatible with the commutation relations (11.1); in any particular realization of those commutation relations, it may be that only some j values are present, while others are absent. But if some j value does occur, then all m values in the range,

$$m = -j, -j+1, \ldots, +j, \tag{11.25}$$

also occur, for if any one m value in this list occurs, i.e., if a nonzero eigenket $|jm\rangle$ exists, then all other nonzero eigenkets with the same j value but other m values in the range (11.25) can be generated from the given one by raising and lowering operators. Thus, the eigenvalue $j(j+1)\hbar^2$ of J^2 is $(2j+1)$-fold degenerate.

Since by assumption the simultaneous eigenkets of J^2 and J_3 are nondegenerate, we must have

$$\begin{aligned} J_+|jm\rangle &= c|j, m+1\rangle, \\ J_-|jm\rangle &= c'|j, m-1\rangle, \end{aligned} \tag{11.26}$$

where c, c' are complex numbers. These numbers can be determined to within a phase by squaring both sides,

$$\begin{aligned} \langle jm|J_-J_+|jm\rangle &= |c|^2 = \hbar^2(j-m)(j+m+1), \\ \langle jm|J_+J_-|jm\rangle &= |c'|^2 = \hbar^2(j+m)(j-m+1). \end{aligned} \tag{11.27}$$

To determine the phases of c, c', we first choose an arbitrary phase convention for the stretched state $|jj\rangle$, and then link the phases of $|jm\rangle$ for $m < j$ to that of $|jj\rangle$ by using lowering operators and demanding that c' be real and positive. Having done this, we can raise the states back up with raising operators, and since the product J_+J_- is nonnegative definite, we find that c is also real and positive. Thus we obtain,

$$\begin{aligned} J_+|jm\rangle &= \hbar\sqrt{(j-m)(j+m+1)}\,|j, m+1\rangle, \\ J_-|jm\rangle &= \hbar\sqrt{(j+m)(j-m+1)}\,|j, m-1\rangle. \end{aligned} \tag{11.28}$$

These phase conventions are standard in the theory of angular momentum and rotations, but of course there is no physics in such conventions.

Now let us go back and worry about degeneracies, and write $|\alpha jm\rangle$ for the simultaneous eigenkets of J^2 and J_3. As usual, we can think of α as an index for an arbitrarily chosen orthonormal basis in the degenerate eigenspaces of J^2 and J_3; at first we imagine that a different, arbitrary choice of such basis vectors is made in every different degenerate eigenspace (i.e., for different values of j, m). We may also imagine that the dimensionalities of these subspaces are different for different values of j, m, i.e., that α has the range, $\alpha = 1, \ldots, N_{jm}$, where N_{jm} is the order of the degeneracy of subspace jm. But in fact it is easy to see that the order of these degeneracies must be independent of m. For let us suppose that the dimensionality of the stretched eigenspace $(jm) = (jj)$ is N_{jj}. Then by applying the lowering operator J_- to the linearly independent states $|\alpha jj\rangle$, $\alpha = 1, \ldots, N_{jj}$, we obtain a set of N_{jj} linearly independent kets in the $(jm) = (j, j-1)$ eigenspace. Therefore the dimensionality of the latter eigenspace is at least N_{jj}. Suppose it is $N_{j,j-1} \geq N_{jj}$. Then we have $N_{j,j-1}$ basis kets $|\alpha j, j-1\rangle$ in the eigenspace $(jm) = (j, j-1)$, which we can raise with J_+ to obtain $N_{j,j-1}$ linearly independent kets in the $(jm) = (jj)$ subspace. Therefore $N_{j,j-1} \leq N_{jj}$, which is consistent only if $N_{j,j-1} = N_{jj}$. Repeating this argument, we see that all the eigenspaces (jm) for given value of j but for m in the range (11.25) have the same dimensionality, which can depend only on j. We denote this dimensionality by N_j, which we call the *multiplicity* of the given j value. The multiplicity can take on any value from 0 (in which case the j value does not occur) to ∞.

As for the eigenkets, they have the form,

$$|\alpha jm\rangle, \quad \alpha = 1, \ldots, N_j, \quad m = -j, \ldots, +j, \tag{11.29}$$

and satisfy the orthogonormality relations,

$$\langle \alpha' j' m' | \alpha jm\rangle = \delta_{\alpha\alpha'}\, \delta_{jj'}\, \delta_{mm'}. \tag{11.30}$$

We will say that these eigenkets $|\alpha jm\rangle$ form the *standard basis* in any space on which some representation of the angular momentum commutation relations acts.

Since the eigenspaces for fixed j but different values of m are linked by raising and lowering operators, it does not make sense to choose an arbitrary orthonormal basis (indexed by α) separately for each one, but rather to choose such a basis in one of the eigenspaces, say, the stretched one $(jm) = (jj)$, and then to define the orthonormal basis in the other eigenspaces by applying lowering operators to the stretched basis. With this understanding, the relations (11.28) become

$$\begin{aligned} J_+|\alpha jm\rangle &= \hbar\sqrt{(j-m)(j+m+1)}\, |\alpha j, m+1\rangle, \\ J_-|\alpha jm\rangle &= \hbar\sqrt{(j+m)(j-m+1)}\, |\alpha j, m-1\rangle. \end{aligned} \tag{11.31}$$

From these and earlier relations, we can write down the matrix elements of the raising and lowering operators, as well as those of J^2 and J_3, in the basis $|\alpha jm\rangle$. These are

$$\begin{aligned} \langle\alpha' j' m'|J_3|\alpha jm\rangle &= m\hbar\,\delta_{\alpha'\alpha}\,\delta_{j'j}\,\delta_{m'm}, \\ \langle\alpha' j' m'|J_+|\alpha jm\rangle &= \hbar\sqrt{(j-m)(j+m+1)}\,\delta_{\alpha'\alpha}\,\delta_{j'j}\,\delta_{m',m+1}, \\ \langle\alpha' j' m'|J_-|\alpha jm\rangle &= \hbar\sqrt{(j+m)(j-m+1)}\,\delta_{\alpha'\alpha}\,\delta_{j'j}\,\delta_{m',m-1}, \\ \langle\alpha' j' m'|J^2|\alpha jm\rangle &= \hbar^2 j(j+1)\,\delta_{\alpha'\alpha}\,\delta_{j'j}\,\delta_{m'm}. \end{aligned} \tag{11.32}$$

The matrix elements of J_1 and J_2 follow trivially from those of $J_\pm$ by

$$J_1 = \frac{1}{2}(J_+ + J_-), \qquad J_2 = \frac{1}{2i}(J_+ - J_-). \tag{11.33}$$

The matrix elements (11.32) are diagonal in j and α, and furthermore depend only on j and m but not on α. They follow directly from the angular momentum commutation relations (11.1) (and on some phase conventions), but they do not depend in any way on the specific nature of the operators which satisfy these commutation relations. Thus, the same matrix elements apply to spin, orbital angular momentum, isospin, etc.

We have now completly solved the problem of finding all possible representations of operators **J** which satisfy the commutation relations (11.1), for we have shown that there exists a basis (the standard basis $|\alpha jm\rangle$) in which the matrix elements of (J_1, J_2, J_3) are given by Eqs. (11.32). It is useful to visualize the matrices representing the angular momentum operators in this basis. To do this, we order the basis kets $|\alpha jm\rangle$ in such a way that m varies most rapidly, α next most rapidly, and j least rapidly. Then the matrices are block-diagonal, beginning with N_0 copies of the 1×1 matrices corresponding to $j=0$, followed by $N_{1/2}$ copies of the 2×2 matrices corresponding to $j=\frac{1}{2}$, etc. For example, if

$N_0 = 3$, $N_{1/2} = 1$, $N_1 = 2$, etc., then the matrices will have the following structure:

$$\langle \alpha' j' m' | X | \alpha j m \rangle = \begin{pmatrix} \boxed{0} & & & & & & \\ & \boxed{0} & & & & & \\ & & \boxed{0} & & & & \\ & & & \boxed{\tfrac{1}{2}} & & & \\ & & & & \boxed{1} & & \\ & & & & & \boxed{1} & \\ & & & & & & \ddots \end{pmatrix}. \qquad (11.34)$$

Here X stands for any of the operators, (J_1, J_2, J_3), or for any function of these operators, such as $J_\pm$, J^2, or, most notably, the rotation operators $\exp(-i\theta\hat{\mathbf{n}} \cdot \mathbf{J}/\hbar)$. In all cases, all copies of the $j = 0$ matrices are identical, all copies of the $j = \frac{1}{2}$ matrices are identical, etc., since their matrix elements depend on m and j but not α. The matrices which go into the blocks on the diagonal depend on which operator X is referred to, but the block structure itself is the same for all choices of X.

Whenever a matrix representing an operator has a block-diagonal form, it means that the operator possesses invariant subspaces; this is the abstract meaning of the block-diagonal structure. A subspace is considered *invariant* under the action of an operator if every vector in that subspace is mapped into another vector in the same subspace by the operator. In the present case, the invariant subspaces are the those spanned by the basis kets $|\alpha jm\rangle$ for fixed values of α and j but variable m, since these are the values of (αjm) which correspond to one of the blocks. For example, the second $j = 1$ block (a 3×3 block) in Eq. (11.34) corresponds to $\alpha = 2$, $j = 1$, $m = 1, 0, -1$. Any linear combination of the basis vectors $|\alpha jm\rangle$ for these three values of (αjm) is mapped into another such linear combination by any function of the angular momentum operators.

Let us denote the invariant subspaces by $\mathcal{E}_{\alpha j}$, so that

$$\mathcal{E}_{\alpha j} = \mathrm{span}\{|\alpha jm\rangle \,|\, -j \le m \le j\}, \qquad (11.35)$$

and so that

$$\dim \mathcal{E}_{\alpha j} = 2j + 1. \qquad (11.36)$$

It is a fact, not proven here but not hard to prove, that the subspaces $\mathcal{E}_{\alpha j}$ possess no smaller subspaces which are invariant under all components of $\mathbf{J}$. Thus, these subspaces are invariant subspaces of minimal dimensionality, in a sense. On account of this property, these subspaces are called *irreducible invariant subspaces.* We shall call them *irreducible subspaces* for short, and for our purposes their most important property is that they are invariant under the action of the rotation operators.

Whenever an operator has an invariant subspace, it is possible to restrict that operator to the subspace (as discussed earlier in Notes 1). For example, if we choose a basis in the invariant subspace, the operator restricted to the subspace becomes represented by a matrix whose size is the dimensionality of the subspace. In the present case, any of the operators X (any function of the angular momentum operators) can be restricted to the irreducible subspaces $\mathcal{E}_{\alpha j}$, whereupon it becomes represented by a $(2j+1)\times(2j+1)$ matrix, and in the standard basis the components of this matrix are independent of α. These are the matrices sitting on the diagonal in Eq. (11.34). In the case that X stands for the components of $\mathbf{J}$, we obtain Hermitian matrices representing $\mathbf{J}$ which satisfy the angular momentum commutation relations (11.1); these matrices are said to form an *irreducible representation* of those commutation relations. In the case that X stands for the rotation operators $\exp(-i\theta\hat{\mathbf{n}}\cdot\mathbf{J}/\hbar)$, we obtain unitary matrices which reproduce the multiplication law for rotations, either in the sense of Eq. (10.3) or (10.31); these matrices are said to form an *irreducible representation* of the rotations. In either case, there is a distinct irreducible representation for each value of j. We see that when studying the general problem of the representations of the angular momentum commutation relations or the rotation operators, it suffices to study the irreducible representations because an arbitrary representation consists of copies of irreducible representations as illustrated by Eq. (11.34).

Let us restrict the angular momentum operators to a single irreducible subspace, and record the matrix elements. We can suppress the index α when writing the basis vectors of the irreducible subspace, since α is constant on such a subspace and the matrix elements do not depend on α. Thus, we write these basis vectors as $|jm\rangle$. Furthermore, j is fixed in a single irreducible subspace, and only m varies. The matrix elements themselves are simple transcriptions of Eq. (11.32):

$$
\begin{aligned}
\langle jm'|J_3|jm\rangle &= m\hbar\,\delta_{m'm},\\
\langle jm'|J_+|jm\rangle &= \hbar\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1},\\
\langle jm'|J_-|jm\rangle &= \hbar\sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1},
\end{aligned}
$$

$$\langle jm'|J^2|jm\rangle = \hbar^2 j(j+1)\,\delta_{m'm} \tag{11.37}$$

These matrix elements do not depend on which irreducible subspace we work with, since they are independent of α. Nor do they depend on the details of the physical interpretation of the operators $\mathbf{J}$ (orbital angular momentum, spin angular momentum, isospin, etc.). They are universal matrices applying to any problem involving angular momenta, and depend only on the commutation relations (11.1) plus the various conventions we have established.

Let us display some examples of the irreducible representations of the angular momentum operators. We will content ourselves with the matrices representing J_3 and J_+, since the matrix for J_- is the Hermitian conjugate of that for J_+, and the matrices for J_1 and J_2 can be obtained from Eq. (11.33). Nor do we bother with J^2, since by Eq. (11.37), its matrix representation on an irreducible subspace is a multiple of the identity.

First, in the case $j = 0$, we have

$$[J_z] = \hbar\,(\,0\,)\,, \tag{11.38}$$

and

$$[J_+] = \hbar\,(\,0\,)\,. \tag{11.39}$$

In this case, the indices m, m' take on the single value 0, and all three components of $\mathbf{J}$ are represented by the 1×1 matrix containing the single element 0.

In the case $j = \frac{1}{2}$, we have

$$[J_z] = \hbar \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{11.40}$$

and

$$[J_+] = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{11.41}$$

Here and below we order the m values from largest to smallest as we move across rows or down columns, so that, for example, the upper right corner of these matrices correspond to $m = \frac{1}{2}$, $m' = -\frac{1}{2}$. These matrices for the case $j = \frac{1}{2}$ are of course equivalent to $\mathbf{J} = (\hbar/2)\boldsymbol{\sigma}$.

In the case $j = 1$, we have

$$[J_z] = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{11.42}$$

and

$$[J_+] = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}. \tag{11.43}$$

Finally, for $j = 3/2$, we have

$$[J_z] = \hbar \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}, \tag{11.44}$$

and

$$[J_+] = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11.45}$$

In all these cases, the matrix for J_3 is diagonal, naturally because we are using an eigenbasis of the operator J_3. The matrices for $J_\pm$ are nonzero only on one diagonal above or below the main diagonal, and are real. Therefore by Eq. (11.33), the matrix for J_1 is real and that for J_2 is pure imaginary. As mentioned, the matrix for J^2 is a multiple of the identity.

When we exponentiate the angular momentum operators in accordance with Eq. (11.2), we obtain the rotation operators. Similarly, when we exponentiate the irreducible matrix representations of the angular momentum operators, we obtain the irreducible representations of the rotation operators. These can be expressed in axis-angle form or in Euler angle form. The symbol D is a standard notation for these matrices, standing for German *Drehung* ("rotation"), or perhaps for German *Darstellung* ("representation"):

$$D^j_{mm'}(\hat{\mathbf{n}}, \theta) = \langle jm|U(\hat{\mathbf{n}}, \theta)|jm'\rangle, \tag{11.46}$$

or

$$D^j_{mm'}(\alpha, \beta, \gamma) = \langle jm|U(\alpha, \beta, \gamma)|jm'\rangle. \tag{11.47}$$

We note in particular that rotations about the z-axis are especially simple in the basis we have chosen, because the matrices are diagonal:

$$D^j_{mm'}(\hat{\mathbf{z}}, \theta) = \langle jm|e^{-i\theta J_z/\hbar}|jm'\rangle = e^{-im\theta}\,\delta_{mm'}. \tag{11.48}$$

This means that two of the factors in the Euler angle representation of the rotation operators [see Eq. (9.42)] are diagonal, so that

$$\begin{aligned} D^j_{mm'}(\alpha, \beta, \gamma) &= \langle m|e^{-i\alpha J_z/\hbar}\, e^{-i\beta J_y/\hbar}\, e^{-i\gamma J_z/\hbar}|m'\rangle \\ &= \sum_{m_1, m_2} \langle m|e^{-i\alpha J_z/\hbar}|m_1\rangle\langle m_1|e^{-i\beta J_y/\hbar}|m_2\rangle\langle m_2|e^{-i\gamma J_z/\hbar}|m'\rangle \\ &= \sum_{m_1, m_2} e^{-i\alpha m_1}\,\delta_{mm_1}\langle m_1|e^{-i\beta J_y/\hbar}|m_2\rangle e^{-i\gamma m'}\,\delta_{m_2 m'} \\ &= e^{-i\alpha m - i\gamma m'}\, d^j_{mm'}(\beta), \end{aligned} \tag{11.49}$$

where

$$d^j_{mm'}(\beta) = \langle m|e^{-i\beta J_y/\hbar}|m'\rangle. \tag{11.50}$$

In Eq. (11.49), we only sum over m in the resolution of the identity, because we are working on a single irreducible subspace. The matrix $d^j_{mm'}(\beta)$ is called the *reduced* rotation matrix; we see that in the Euler angle decomposition of an arbitrary rotation, only the rotation about the y-axis is nontrivial, and it depends only on the one Euler angle β. Furthermore, since the matrix elements of J_2 are purely imaginary under our conventions, the reduced matrix elements $d^j_{mm'}(\beta)$ are purely real. This is one of the conveniences of the zyz-convention for Euler angles in quantum mechanics.

Therefore when tabulating the irreducible matrix representations of the rotation operators in Euler angle form, it suffices to tabulate only the reduced rotation matrices. The first few of these are easy to work out, and for more complicated cases, there exist tables or explicit formulas. For the case $j = 0$, the result is trivial:

$$d^0_{mm'}(\beta) = (\,1\,)\,. \tag{11.51}$$

We see that a rotation does nothing to a system of zero angular momentum, such as a spin-0 particle. For the case $j = 1/2$, we use the properties of the Pauli matrices to obtain

$$d^{1/2}_{mm'}(\beta) = \cos(\beta/2) - i\sigma_y \sin(\beta/2) = \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}. \tag{11.52}$$

For the case $j = 1$, we can find recursions among the powers of the matrix for J_y, and sum the exponential series to obtain

$$d^1_{mm'}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\sin\beta/\sqrt{2} & \frac{1}{2}(1-\cos\beta) \\ \sin\beta/\sqrt{2} & \cos\beta & -\sin\beta/\sqrt{2} \\ \frac{1}{2}(1-\cos\beta) & \sin\beta/\sqrt{2} & \frac{1}{2}(1+\cos\beta) \end{pmatrix}. \tag{11.53}$$

This calculation is repeated in Sakurai.

The matrix elements for J_3, given by Eq. (11.48), contain an important lesson. Since m is integral (half-integral) when j is integral (half-integral), and since the phases $e^{-im\theta}$ occur on the diagonal of the matrix elements for J_3, we see that the irreducible representations of the rotation operators form a double-valued representation of $SO(3)$ in the case of half-integral j, and a single-valued representation in the case of integral j. In all cases, the D-matrices form a (proper, single-valued) irreducible representations of the group $SU(2)$.

The D-matrices have many properties, of which we mention only one here. If U is a rotation operator and $D^j_{mm'}(U)$ the corresponding matrix, then the operator U^{-1} corresponds to the matrix D^{-1}. But since U is unitary, so is the matrix D, and we have

$$D^j_{mm'}(U^{-1}) = [D^j(U)^{-1}]_{mm'} = [D^j(U)^\dagger]_{mm'} = D^{j*}_{m'm}(U). \tag{11.54}$$

Often in quantum mechanics we need to find the action of a rotation operator on some state. When the state is expanded in terms of the standard basis, the problem is equivalent to rotating a vector $|\alpha jm\rangle$ of the standard basis. That is, we seek an expression for $U|\alpha jm\rangle$, where U is a rotation operator. But since the irreducible subspaces are invariant under rotations, the vector $U|\alpha jm\rangle$ must be expressible as a linear combination of other basis vectors in the same irreducible subspace, i.e.,

$$U|\alpha jm\rangle = \sum_{m'} |\alpha jm'\rangle\langle\alpha jm'|U|\alpha jm\rangle = \sum_{m'} |\alpha jm\rangle\langle jm'|U|jm\rangle. \tag{11.55}$$

Notice that in the first sum we have effectively introduced a resolution of the identity, but only in the irreducible subspace. That is, there is no sum over α or j, because U is diagonal in α and j, which is the same as saying that the irreducible subspace is invariant under U. In the second sum we have suppressed the α indices in the matrix elements of U, since these matrix elements do not depend on α; the resulting matrix element is just a component of a D matrix. Thus, we have

$$\boxed{U|\alpha jm\rangle = \sum_{m'} |\alpha jm'\rangle D^j_{m'm}(U),} \tag{11.56}$$

which is often useful. Notice the positions of the indices m', m in this formula.

We consider one final topic in the theory of rotation operators, namely, the generalized adjoint formula. This topic is somewhat disjoint from the rest of the material in these notes, since it does not depend on the theory of irreducible representations.

We recall that we derived a version of the adjoint formula for classical rotations in Eq. (9.31), and later we found an analogous formula, Eq. (10.24), for spin-$\frac{1}{2}$ rotations. We now generalize this to arbitrary representations of the rotation operators. The generalization is obvious; it is

$$\boxed{U\mathbf{J}U^\dagger = \mathsf{R}^{-1}\mathbf{J},} \tag{11.57}$$

where $U = U(\hat{\mathbf{n}}, \theta)$ and $\mathsf{R} = \mathsf{R}(\hat{\mathbf{n}}, \theta)$. Notice that the left hand side is quadratic in U, so in the case of double-valued representations of $SO(3)$, it does not matter which U operator we choose to represent the rotation R.

To prove Eq. (11.57), we define the operator vector,

$$\mathbf{X}(\theta) = U(\hat{\mathbf{n}}, \theta)\,\mathbf{J}\,U(\hat{\mathbf{n}}, \theta)^\dagger, \tag{11.58}$$

and we note the initial condition,

$$\mathbf{X}(0) = \mathbf{J}. \tag{11.59}$$

Next we obtain a differential equation for $\mathbf{X}(\theta)$:

$$\frac{d\mathbf{X}(\theta)}{d\theta} = \frac{dU}{d\theta}\mathbf{J}U^\dagger + U\mathbf{J}\frac{dU^\dagger}{d\theta} = -\frac{i}{\hbar}U[\hat{\mathbf{n}}\cdot\mathbf{J},\mathbf{J}]U^\dagger$$
$$= -\hat{\mathbf{n}}\times(U\mathbf{J}U^\dagger) = -(\hat{\mathbf{n}}\cdot\mathsf{J})\mathbf{X}. \tag{11.60}$$

The solution is

$$\mathbf{X}(\theta) = \exp\bigl(-\theta\hat{\mathbf{n}}\cdot\mathsf{J}\bigr)\mathbf{X}(0) = \mathsf{R}(\hat{\mathbf{n}},\theta)^{-1}\mathbf{J}, \tag{11.61}$$

which is equivalent to the adjoint formula (11.57).

Finally, we can dot both sides of Eq. (11.57) by $-i\theta\hat{\mathbf{n}}/\hbar$ and exponentiate, to obtain a formula analogous to Eq. (9.33). After placing 0 subscripts on U and R for clarity, the result is

$$U_0 U(\hat{\mathbf{n}},\theta)U_0^\dagger = U(\mathsf{R}_0\hat{\mathbf{n}},\theta), \tag{11.62}$$

where U_0 and R_0 are corresponding quantum and classical rotations. Notice again that the left hand side is quadratic in U_0, so that in the case of double-valued representations it does not matter which U_0 operator we choose to represent the rotation R_0.