## Summary

$$\left.\begin{aligned} t' &= \gamma\left(t - \tfrac{v}{c^2}x\right) \\ x' &= \gamma\,(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \text{ boost in } x\text{-direction} \qquad \left.\begin{aligned} t' &= \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right) \\ \vec{x}' &= \vec{x} + (\gamma-1)\frac{(\vec{v}\cdot\vec{x})\vec{v}}{v^2} - \gamma\vec{v}t \end{aligned}\right\}$$

boost in arb. direc.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}, \quad \vec{\beta} = \frac{\vec{v}}{c}.$$

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Some properties of Lorentz transformation.

More symmetrical in variables

$$\left.\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}\right\}$$

all components have dimensions of length.

superscripts explained later.

or $x^\mu$ for short, $\mu = 0, 1, 2, 3$.

For example, boost in x-direc. has form,

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

Transformation matrix is dimensionless.

Write this in the form, $\boxed{x'^\mu = L^\mu{}_{\cdot\nu}\, x^\nu}$ (summation convention) where

$$L^\mu{}_{\cdot\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all 4 coordinates.

Explain upper, lower posn's of $\mu, \nu$ on $L^\mu{}_{\cdot\nu}$ later, just note for now that 1st index (superscript) is the row, 2nd index (subscript) is the column.

Similarly can write out matrix $L^{\mu}_{\cdot\,\nu}$ for a boost in arbitrary direction.

One of postulates that went into L.T. was constancy of speed of light. Thus, the eqn. of a spherical light pulse must be the same in both frames:

$$c^2t^2 - |\vec{x}|^2 = 0 \qquad \Rightarrow \qquad c^2t'^2 - |\vec{x}'|^2 = 0.$$

Actually, the L.T. satisfies a stronger condition,

$$c^2t^2 - |\vec{x}|^2 = c^2t'^2 - |\vec{x}'|^2$$

for all points (events) of space time with coordinates $(\vec{x},t)$ and $(\vec{x}',t')$ in two inertial frames. This is more conveniently stated by introducing the tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ 0 & & -1 & \\ & & & -1 \end{pmatrix},$$

so that

$$x^{\mu} g_{\mu\nu} x^{\nu} = x'^{\mu} g_{\mu\nu} x'^{\nu}.$$

This is the condition we claim all L.T.'s satisfy. It can be proven from the original postulates, or directly from the results (the L.T.'s we found). Substitute $x'^{\mu} = L^{\mu}_{\cdot\,\alpha} x^{\alpha}$, and we get

$$x^{\mu} g_{\mu\nu} x^{\nu} = x^{\alpha} L^{\mu}_{\cdot\,\alpha} g_{\mu\nu} L^{\nu}_{\cdot\,\beta} x^{\beta} \qquad \text{(must hold for all } x^{\alpha})$$

or

$$g_{\alpha\beta} = L^{\mu}_{\cdot\,\alpha} g_{\mu\nu} L^{\nu}_{\cdot\,\beta}$$

or $\boxed{g = L^{t} g L}$ as matrices

To verify this property, we may just multiply matrices:

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ check for boost in x-direc.

And can check for boost in arb. direction.

This is a fundamental invariance rule for Lorentz transformations; it holds also for coordinate differences $\Delta x^\mu$ between 2 events ($\Delta x'^\mu$ in another inertial frame), or infinitesimal coordinate differences $dx^\mu$:

$$\underset{\substack{\uparrow \\ \text{call it } ds^2.}}{ds^2} = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu}\, dx'^\mu dx'^\nu .$$

or $\Delta s^2 = g_{\mu\nu}\, \Delta x^\mu \Delta x^\nu$ etc.

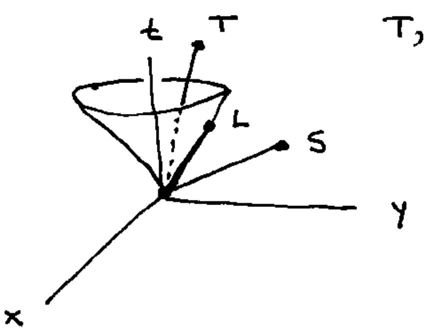
Since $\Delta s^2$ is the same in all inertial frames (all observers), it provides an invariant characterization of an interval:

Let $\Delta x^\mu = x_1^\mu - x_0^\mu$ diff. betw. 2 events 0 and 1.

Then

$$\left.\begin{array}{ll} \Delta s^2 > 0 & \text{time-like} \\ \Delta s^2 = 0 & \text{light-like} \\ \Delta s^2 < 0 & \text{space-like} \end{array}\right\} \text{interval.}$$

Put $x_0^\mu = 0$, draw light cone:




T, L, S = time-, light-, space-like intervals.

Light cone itself is a Lorentz invariant.

$ds^2 > 0$

A <u>time-like</u> interval always lies inside the light cone (fwd or bkwd). You can tell which one by the sign of $\Delta t$, which is the same for all observers. If an interval $\Delta x^\mu$ is time-like, then there always exists a frame in which $\Delta x^\mu = (\Delta t, 0)$ ie for which $\Delta \vec{x} = 0$. All observers agree on the temporal ordering.

$ds^2 = 0$

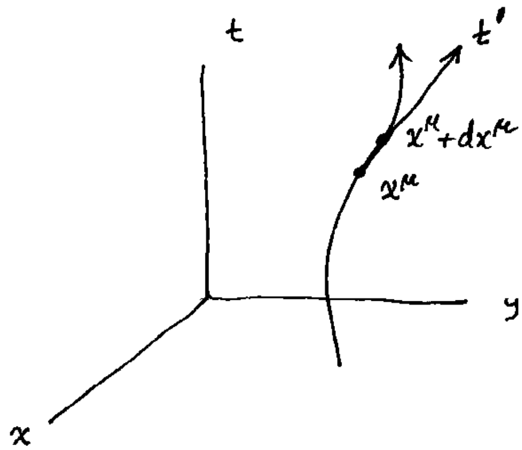
A <u>light-like</u> interval connects two events that can be connected by a light signal.

$ds^2 < 0$

A <u>space-like</u> interval always lies outside the light cone. If $\Delta x^\mu$ is space-like, there always exists a frame in which $\Delta x^\mu = (0, \Delta\vec{x})$, ie, $\Delta t = 0$. In that frame, the 2 events are simultaneous. There also exist frames for which $\Delta t > 0$ or $\Delta t < 0$, ie the temporal ordering of the events depends on the observer.

Time-like, light-like intervals are <u>causally connected</u>; space-like intervals are <u>causally disconnected</u>.

Consider 2 events infinitesimally separated on the world line of a particle:




The interval $dx^\mu$ is certainly time-like, since $v = \frac{d|\vec{x}|}{dt} < c$

$\Rightarrow \quad c^2 dt^2 - |d\vec{x}|^2 > 0$. Thus there is a frame (call it primed frame) in which $d\vec{x}' = 0$. This is the <u>rest frame</u> of the particle at that instant, since

$$\frac{d\vec{x}'}{dt'} = 0.$$

The $t'$-axis is <u>tangent</u> to the world line.

In fact, in the rest-frame, $c\,dt' = ds$. Give it another name, call it $c\,d\tau$, it is the elapsed time on a clock carried by the particle. Thus, in any frame,

$$c\,d\tau = \sqrt{c^2dt^2 - |d\vec{x}|^2} = c\,dt\sqrt{1 - v^2/c^2}$$

$$d\tau = \frac{dt}{\gamma}.$$

The factor $\frac{1}{\gamma}$ expresses the time dilation. Thus the elapsed time along a segment of the orbit, as measured by a clock carried with the particle, is

$$\tau = \int_{\substack{\text{world}\\ \text{line}}} \frac{dt}{\gamma} = \underline{\text{proper time}}$$

Proper time can be used as a parameter to describe the trajectory of a massive particle. So can $t$ (coordinate time), but that is less convenient because it depends on the observer. For photons, proper time cannot be used as a parameter (since $d\tau = 0$ for light-like intervals.)

Using proper time $\tau$ to parameterize a world line is like using arc-length to parameterize a curve in 3D space.

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The equation
$$L^t g L = g$$
for Lorentz transformations may be compared to
$$R^t I R = I$$
for ordinary rotations ($I$ = identity, $R$ = orthogonal). Ordinary rotations preserve the Euclidean metric of 3D space ($I$) while Lorentz trasformations preserve the indefinite <u>Minkowski</u> metric $g$ of 4D space-time.

An ordinary rotation in 2D can be written in terms of an angle:

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R(\theta)$$

Sim., a Lorentz transformation in $(x,t)$ can be written in terms of a hyperbolic angle:

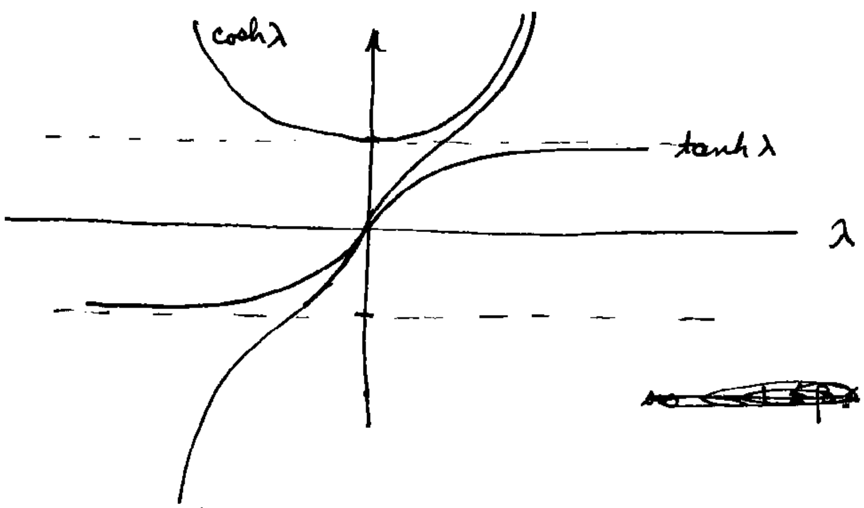
$$L = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix} = L(\lambda).$$

↑ like a rotation in hyperbolic angles.

where

$$\left.\begin{aligned} \gamma &= \cosh\lambda \\ \gamma\beta &= \sinh\lambda \\ \beta &= \tanh\lambda \end{aligned}\right\}.$$

$\lambda$ is called the <u>rapidity</u>.




Ranges:

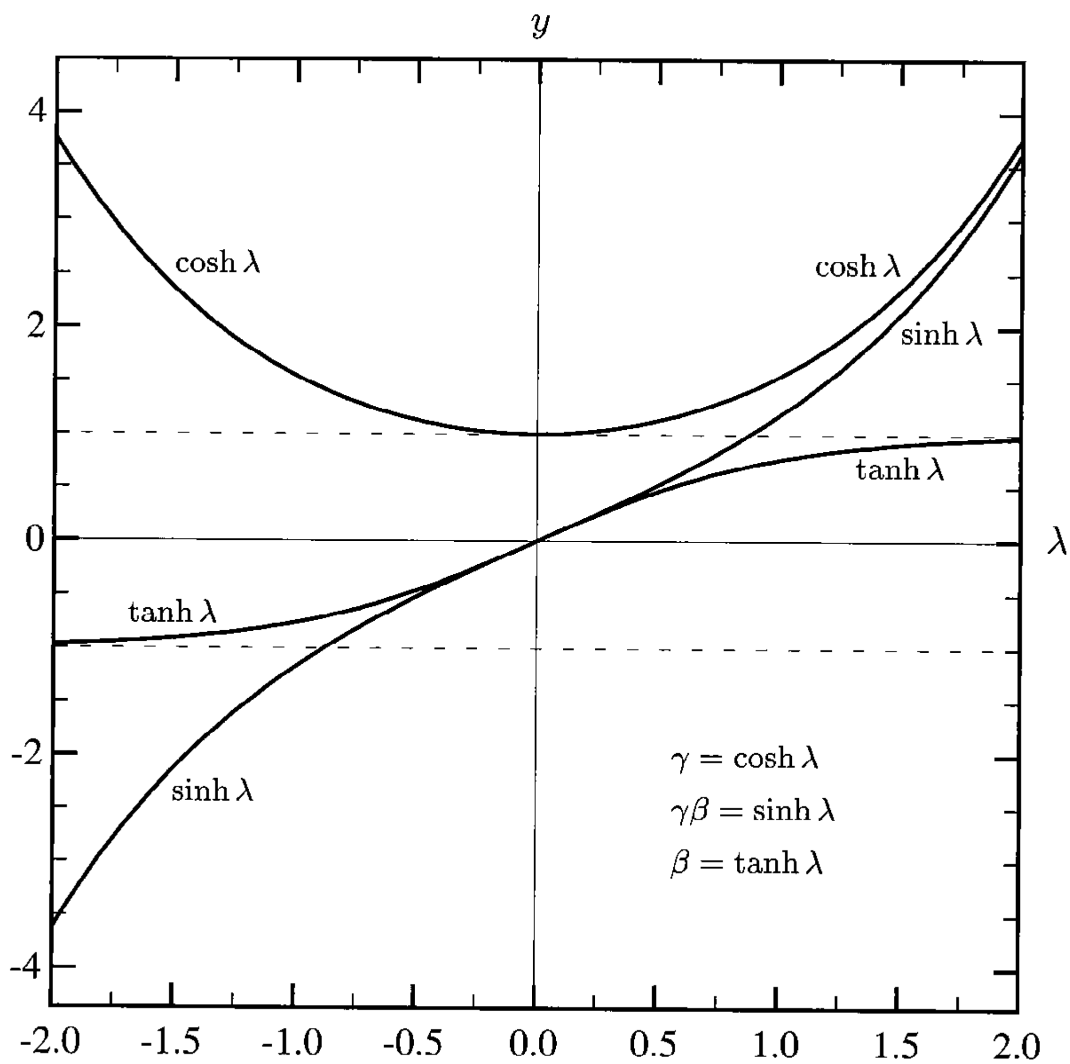
$$-c < v < +c$$
$$-1 < \beta < +1$$
$$-\infty < \lambda < +\infty.$$

Just like $R(\theta_1) R(\theta_2) = R(\theta_1+\theta_2)$

we have $\cancel{R(\lambda_1) R(\lambda_2) = R(\lambda_1+\lambda_2)}$ from trig identities

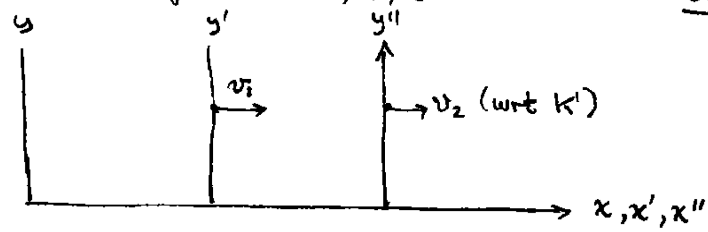
$$L(\lambda_1) L(\lambda_2) = L(\lambda_1+\lambda_2)$$

Better plot of $\gamma, \gamma\beta, \beta$ as functions of rapidity $\lambda$.



## Addition of velocities

Where this occurs: 3 frames $K, K', K''$




Let $K'$ be moving with velocity $v_1$ wrt $K$
" $K''$ " " " " $v_2$ " $K'$.

What is velocity of $K''$ wrt $K$? Use rapidity

$$\begin{pmatrix} x^{0''} \\ x^{1''} \end{pmatrix} = L(\lambda_2)\begin{pmatrix} x^{0'} \\ x^{1'} \end{pmatrix} = \underbrace{L(\lambda_2)L(\lambda_1)}_{= L(\lambda_1+\lambda_2).}\begin{pmatrix} x^{0} \\ x^{1} \end{pmatrix}$$

So $\lambda = \lambda_1 + \lambda_2$.

$$\frac{v}{c} = \tanh\lambda = \tanh(\lambda_1+\lambda_2) = \frac{\tanh\lambda_1 + \tanh\lambda_2}{1+\tanh\lambda_1\tanh\lambda_2}.$$

$$= \frac{\frac{v_1}{c}+\frac{v_2}{c}}{1+\frac{v_1 v_2}{c^2}}, \quad \text{ie,} \quad \boxed{v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}}$$

Note, if $|v_1| < c$ and $|v_2| < c$, then $|v| < c$. Follows most easily from properties of tanh function.