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Summary

$$-\Box f = \nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = -4\pi\, S(\vec{x},t).$$ (inhomogeneous wave equ).

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x},t;\vec{x}',t') = -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')$$

$$G_{\pm}(\vec{x},t;\vec{x}',t') = \frac{\delta\left(t-t' \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}$$

upper sign: retarded
lower : advanced

$$f_{\pm}(\vec{x},t) = \int d^3\vec{x}' \; \frac{S\left(\vec{x}', t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}.$$

$$-\Box\Phi = -4\pi\rho - \frac{1}{c}\frac{\partial L}{\partial t}$$
$$-\Box\vec{A} = -\frac{4\pi}{c}\vec{J} + \nabla L$$
$$L = \nabla A + \frac{1}{c}\frac{\partial\Phi}{\partial t}.$$

$$\left[S(\vec{x}',t')\right]_{ret} = S\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$\left[S(\vec{x}',t')\right]_{adv} = S\left(\vec{x}', t + \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

Lorenz gauge,

$$\Box\Phi = 4\pi\rho$$
$$\Box\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$$ (defn of Lorenz gauge)

$$\left.\begin{aligned}
\Phi(\vec{x},t) &= \int d^3\vec{x}' \; \frac{[\rho(\vec{x}',t')]_{ret}}{|\vec{x}-\vec{x}'|} \\
\vec{A}(\vec{x},t) &= \frac{1}{c}\int d^3\vec{x}' \; \frac{[\vec{J}(\vec{x}',t')]_{ret}}{|\vec{x}-\vec{x}'|}
\end{aligned}\right\}$$ starting point for solution of all radiation problems.

A second gauge commonly used is <u>Coulomb gauge</u>:

$$\boxed{\nabla\cdot\vec{A}=0} \qquad \text{(defn of Coulomb gauge).}$$

First, easily show that Coulomb gauge exists: Let $\vec{A}'$ = any gauge, where $\nabla\cdot\vec{A}'\neq 0$. Then with $\vec{A}=\vec{A}'+\nabla\Lambda$, we have

$$\nabla\cdot\vec{A}=\nabla\cdot\vec{A}'+\nabla^2\Lambda,$$

so solve $\nabla^2\Lambda=-(\nabla\cdot\vec{A}')$. (Poisson eqn).

Use free space G-fn, get Λ, do gauge xfm, and $\nabla\cdot\vec{A}=0$.

Coulomb gauge simplifies the eqns of evolution for the potentials:

$$\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-4\pi\rho-\frac{1}{c}\frac{\partial L}{\partial t}$$

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\frac{4\pi}{c}\vec{J}+\nabla L$$

$$L=\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}.$$

So in Coulomb gauge,

$$\boxed{\begin{aligned}&\nabla^2\Phi=-4\pi\rho\\ &\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\frac{4\pi}{c}\vec{J}+\frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi\end{aligned}} \leftarrow \text{Poisson eqn.}$$

Now Φ eqn decoupled from $\vec{A}$ eqn, solve it first. Use <u>Poisson</u> Green's fn, get

$$\Phi(\vec{x},t)=\int d^3x'\;\frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

just like in electrostatics, but notice everything here is t-dep, and there is <u>no</u> <u>retardation</u> in Φ. Have to interpret physically.

But when Φ is known, plug into $\vec{A}$ eqn and use t-dep. (retarded) G-fn to solve for $\vec{A}$.

Discuss advantages, disadvantages of Lorenz, Coulomb gauge.

Concepts of <u>transverse</u>, <u>longitudinal</u> fields clarifies Coulomb gauge.
Digression. Let $\vec{F}(\vec{x})$ be a vector field. Define Fourier transform by

$$\tilde{\vec{F}}(\vec{k}) = \int d^3\vec{x}\; e^{-i\vec{k}\cdot\vec{x}}\, \vec{F}(\vec{x})$$

$$\vec{F}(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\; e^{+i\vec{k}\cdot\vec{x}}\, \tilde{\vec{F}}(\vec{k}).$$

$\vec{F}$ may have t-dep, but ignore it.

Terms <u>transverse</u> <u>longitudinal</u> refer to $\vec{k}$-space:

$\vec{F}$ is transverse if $\vec{k}\cdot\tilde{\vec{F}}(\vec{k}) = 0$. $\quad \tilde{\vec{F}} \perp \vec{k}$

$\vec{F}$ is longitudinal if $\vec{k}\times\tilde{\vec{F}}(\vec{k}) = 0$. $\quad \tilde{\vec{F}} \parallel \vec{k}$.

Note operator relations:

In $\vec{x}$-space	In $\vec{k}$-space
∇	$i\vec{k}$
$-i\vec{x}$	$\partial/\partial\vec{k} \equiv \nabla_{\vec{k}}$.

So, $\nabla\cdot \rightarrow i\vec{k}\cdot$
$\quad\;\; \nabla\times \rightarrow i\vec{k}\times$

Hence:

$\vec{F}$ is transverse if $\nabla\cdot\vec{F} = 0$.
$\vec{F}$ is longitudinal if $\nabla\times\vec{F} = 0$.

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An "arbitrary" vector field can be broken into longitudinal and transverse parts: To get $\perp$ part, for example,

$$\vec{F}(\vec{x}) \xrightarrow{\text{F.T.}} \tilde{\vec{F}}(\vec{k}) \xrightarrow[\perp \text{ to } \vec{k}]{\text{project}} \tilde{\vec{F}}_\perp(\vec{k}) \xrightarrow[\text{F.T.}]{\text{Inverse}} \vec{F}_\perp(\vec{r}).$$

Sim. for longitudinal. Let's do this explicitly:

$$\vec{F}_\perp(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \tilde{\vec{F}}_\perp(\vec{k})$$

$$= \left(I - \frac{\vec{k}\vec{k}}{k^2}\right)\cdot \tilde{\vec{F}}(\vec{k})$$

dyad notation for $\perp$ projector.

$$= \int d^3\vec{x}'\, e^{-i\vec{k}\cdot\vec{x}'}\, \vec{F}(\vec{x}').$$

Put it together,

$$\vec{F}_\perp(\vec{x}) = \int \frac{d^3\vec{k}\, d^3\vec{x}'}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left(I - \frac{\vec{k}\vec{k}}{k^2}\right)\cdot \vec{F}(\vec{x}')$$

or, $$F_{\perp i}(\vec{x}) = \int d^3\vec{x}'\, \underbrace{\Delta^\perp_{ij}(\vec{x}, \vec{x}')}\, F_j(\vec{x}')$$

transverse projection kernel, sometimes called "transverse δ-fn".

$$\Delta^\perp_{ij}(\vec{x}, \vec{x}') = \int \frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}(\vec{x}-\vec{x}')} \cdot \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$$

$$= \text{fn of } \vec{x}-\vec{x}'.$$

Similarly can do long. projection.

Can also do this directly in $\vec{x}$-space. Write...

$$\vec{F}(\vec{x}) = \vec{F}_{\parallel}(\vec{x}) + \vec{F}_{\perp}(\vec{x}),$$

where
$$\nabla \times \vec{F}(\vec{x}) = \nabla \times \vec{F}_{\perp}(\vec{x})$$
$$\nabla \cdot \vec{F}(\vec{x}) = \nabla \cdot \vec{F}_{\parallel}(\vec{x}).$$

So suppose $\vec{F}$ is given, and you want $\vec{F}_{\parallel}$. Then $\nabla \cdot \vec{F}$ known, and $\vec{F}_{\parallel}$ satisfies:

$$\left.\begin{aligned} \nabla \cdot \vec{F}_{\parallel} &= \nabla \cdot \vec{F} = \text{known} \\ \nabla \times \vec{F}_{\parallel} &= 0 \end{aligned}\right\}.$$

So write $\vec{F}_{\parallel} = -\nabla S$, hence $\nabla^2 S = -\nabla \cdot \vec{F}$, solve with free sp. G-fn. for Poisson,

$$S(\vec{x}) = \frac{1}{4\pi} \int d^3\vec{x}' \, \frac{\nabla' \cdot \vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$= -\frac{1}{4\pi} \int d^3\vec{x}' \, \nabla' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \cdot \vec{F}(\vec{x}') \qquad \text{integr. by parts.}$$

$$= + \nabla \cdot \underbrace{\left[\frac{1}{4\pi} \int d^3\vec{x}' \, \frac{\vec{F}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right]} \qquad \begin{array}{c} \nabla = -\nabla' \\ \text{when acting} \\ \text{on fn. of} \\ \vec{x} - \vec{x}' \end{array}$$

So, $$\vec{F}_{\parallel}(\vec{x}) = -\nabla \, \nabla \cdot \left[\quad \right].$$

Similarly, $$\vec{F}_{\perp}(\vec{x}) = \nabla \times \nabla \times \left[\text{ same integral. } \right]$$

An important point is that the projection process is <u>local</u> in $\vec{k}$-space, but <u>non-local</u> in $\vec{x}$-space.

Transverse, longitudinal ideas clarify Coulomb gauge. Some points:

① In Coulomb gauge, $\nabla\cdot\vec{A}=0$ so $\vec{A}$ is <u>purely</u> <u>transverse</u>.
Under general gauge transf., $\vec{A}=\vec{A}'+\nabla\Lambda$. $\nabla\Lambda$ is a purely longitudinal field, so the gauge xfm. changes only <u>longitudinal</u> <u>part</u> of $\vec{A}$. This is the non physical part, we might say. So in Coulomb gauge, we set this part $=0$.

② In any gauge, $\vec{B}=\nabla\times\vec{A}$ = purely transverse $(\nabla\cdot\vec{B}=0)$.
Also,
$$\vec{E} = \underbrace{-\nabla\Phi}_{\text{longitudinal}} \underbrace{-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}}_{\text{transverse, \underline{if} you use Coul. gauge.}}$$

③ Go back to eqns for $\Phi,\vec{A}$: In Coul. gauge

$$\underbrace{\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}}_{\text{purely transverse}} = -\frac{4\pi}{c}\vec{J} + \underbrace{\frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi}_{\text{purely longitudinal.}}$$

So ~~∅~~ RHS must be purely transverse, and $\frac{1}{c}\frac{\partial}{\partial t}\nabla\Phi$ must cancel ~~t~~ long. part of $-\frac{4\pi}{c}\vec{J}$ term.

This is true:

$$\vec{J}_{\parallel}(\vec{x}) = -\nabla\,\nabla\cdot\,\frac{1}{4\pi}\int d^3\vec{x}'\,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= -\,\nabla\,\frac{1}{4\pi}\int d^3\vec{x}'\,\frac{\nabla'\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= +\,\nabla\,\frac{1}{4\pi}\int d^3\vec{x}'\,\frac{\frac{\partial\rho}{\partial t}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= +\,\nabla\frac{\partial}{\partial t}\Phi.$$

So,

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}_{\perp}.}$$

in Coul. gauge, $\vec{A}$ is driven by transverse current.

(4) <u>Final point</u>: The decomposition into Long., transv. fields is <u>not</u> Lorentz invariant.