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New subject, Maxwell eqns and potentials. (Ch. 6).

Begin w. ME's (switch to Gaussian units): (μ-scopic only).

$$\left.\begin{aligned} \nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{E} &= -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\end{aligned}\right\}\ \text{homog.} \quad \Rightarrow \quad \exists\ \Phi, \vec{A} \text{ such that } \left.\begin{aligned}\vec{B} &= \nabla\times\vec{A} \\ \vec{E} &= -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\end{aligned}\right\}$$

$$\left.\begin{aligned} \nabla\cdot\vec{E} &= 4\pi\rho \\ \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &= \frac{4\pi}{c}\vec{J}\end{aligned}\right\}\ \text{inhomog.}$$

not unique, Gauge transformation

$$\vec{A} = \vec{A}' + \nabla\psi$$
$$\Phi = \Phi' - \frac{1}{c}\frac{\partial\psi}{\partial t}$$

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plug potentials into Maxwell eqns, do algebra, find

$$\left.\begin{aligned} -\Box\Phi = \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} &= -4\pi\rho - \frac{1}{c}\frac{\partial L}{\partial t} \\ \overset{\text{minus}}{\Box}\vec{A} = \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{J} + \nabla L\end{aligned}\right\}.$$

where $L \equiv \nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}$.

Have wave or d'Alembertian operator on LHS. $-\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.

~~So~~ Since potentials are not unique, can use choice of gauge to simplify. One obvious choice is

$$L = 0. \qquad \text{(Lorenz gauge)}.$$

Then

$$\left.\begin{aligned} -\Box\Phi &= -4\pi\rho \\ -\Box\vec{A} &= -\frac{4\pi}{c}\vec{J}\end{aligned}\right\}.$$

Can we find Lorenz gauge? Does it exist, ie suppose we have $\vec{A}', \Phi'$ such that $L' \neq 0$. Do a gauge xfm, you get

$$L = L' + \nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \text{ (demand)}. \quad \text{So, } -\Box\psi = -L'.$$

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So if we can solve inhomog. wave eqn, then we can always find ψ to convert arb. gauge $(\Phi', \vec{A}')$ into Lorenz gauge.

Note that soln is not unique: If $\Box \psi = 0$, then ψ converts one choice of Lorenz gauge into another.

So now let's look at inhomog. wave eqn. Want to solve

$$-\Box f(\vec{x},t) = -4\pi S(\vec{x},t)$$

$f =$ some field
$S =$ source.

For this we introduce Green's fn,

(−4π's conventional)

$$-\Box G(\vec{x},t,\vec{x}',t') = -4\pi\, \delta(\vec{x}-\vec{x}')\delta(t-t').$$

Usual statements hold, G not unique, must satisfy bdry conds. etc. But if we find G, then solution is

$$f(\vec{x},t) = \int d^3\vec{x}' \int dt' \quad G(\vec{x},t,\vec{x}',t')\; S(\vec{x}',t').$$

So we need G. Note 1st, because of translational invariance,

$$G(\vec{x},t,\vec{x}',t') = G(\vec{x}-\vec{x}', t-t')$$

G depends only on differences betw. source, field. Then ...

$$-\Box G(\vec{x},t) = -4\pi\, \delta(\vec{x})\delta(t)$$

Now Fourier transform G in time:

$$G(\vec{x},t) = \int \frac{d\omega}{2\pi}\, \tilde{G}(\vec{x},\omega)\, e^{-i\omega t}, \quad \text{plug in,} \quad \frac{\partial^2}{\partial t^2} = -\omega^2,$$

$$\int \frac{d\omega}{2\pi}\, e^{-i\omega t} \left[\nabla^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\vec{x},\omega) = -4\pi\, \delta(\vec{x})\delta(t).$$

apply inverse F.T. (i.e. apply $\int dt\, e^{i\omega' t}$ to both sides)

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$$(\nabla^2 + k^2)\, G_k(\vec{x}) = -4\pi\, \delta(\vec{x}).$$

where $k \equiv \frac{\omega}{c}$, $\quad G_k(\vec{x}) = \tilde{G}(\vec{x}, \omega)$.

Thus $G_k(\vec{x})$ is Green's fn. for Helmholtz eqn, $\quad (\nabla^2 + k^2)\psi = 0$.

~~Now note~~ In fact, we know soln. when $k \to 0$, get Poisson eqn,

$$\lim_{k \to 0} G_k(\vec{x}) = \frac{1}{|\vec{x}|} = \frac{1}{r}.$$

So need to generalize. So note, eqn. is rotationally invariant, soln exists that is fn of r only. $\quad G_k = G_k(r)$. So,

$$\left[\frac{1}{r}\frac{\partial^2}{\partial r^2} r + k^2\right] G_k(r) = -4\pi\, \delta(\vec{x}) = 0 \text{ if } r \neq 0.$$

Solve when $r \neq 0$. Set $F = r G_k$,

$$\frac{d^2 F}{dr^2} + k^2 F = 0. \qquad \text{Harm. osc. eqn.}$$

$$F = A e^{ikr} + B e^{-ikr}.$$

$$G_k(r) = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}.$$

This argument is flawed.

To handle δ-fn at $r=0$, use fact that as $k \to 0$, must get $G = \frac{1}{r}$.
$\Rightarrow A + B = 1$. Get 2 Green's fn, $\quad G^+$: $A=1$, $B=0$ | G^-: $A=0$, $B=1$

$$\boxed{G_k^{\pm}(r) = \frac{e^{\pm ikr}}{r}} \qquad \text{Green's fn. for Helmholtz.}$$

can show by direct substitution that this satisfies Helmholtz eqn. w. RHS $= -4\pi\,\delta(\vec{r})$.

No surprise that we get 2 Green's fns, we know they are not unique. But will have to interpret. Now do inverse FT to get $G(\vec{x},t)$ back.

$$\overset{\pm}{G}(\vec{x},t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \frac{e^{\pm ikr}}{r} \qquad k = \frac{\omega}{c}.$$

$$= \frac{\delta\left(\pm \frac{r}{c} - t\right)}{r}, \quad \text{or,}$$

$$\boxed{G_{\pm}(\vec{x},t,\vec{x}',t') = \frac{\delta\left(\pm \frac{|\vec{x}-\vec{x}'|}{c} - (t-t')\right)}{|\vec{x}-\vec{x}'|}.}$$ 2 Green's fns. for wave eqn.

Note that $e^{i(kr-\omega t)}$ = wave travelling out from origin
$e^{i(-kr-\omega t)}$ = waves " into origin.

Plug in, $\Box^2 f = -4\pi D$, get

$$f(\vec{x},t) = \int d^3\vec{x}'\,dt' \; \frac{\delta\left(t'-t \pm \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\, D(\vec{x}',t')$$

Can do t-integral,

$$= \int d^3\vec{x}' \; \frac{D\left(\vec{x}',\, t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}.$$

Looks like Coulomb's law, in fact it is if source has no t-dep (steady or static source). But when t-dep. present, you have to evaluate source at either retarded or advanced time:

G_+ : retarded time: $t - \frac{|\vec{x}-\vec{x}'|}{c}$ = field time t − light time, source → field.

G_- : advanced time: $t + \frac{|\vec{x}-\vec{x}'|}{c}$ = " + " .