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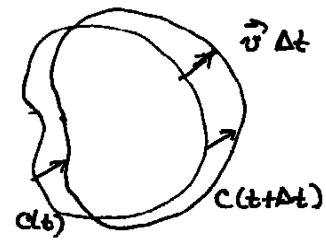
Summary:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}.$$



moving loop:

$$\frac{dF}{dt} = \frac{d}{dt}\int_{S(t)} \vec{B}(t)\cdot d\vec{a} = \frac{1}{\Delta t}\left\{ \int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot d\vec{a} - \int_{S(t)} \vec{B}(t)\cdot d\vec{a} \right\}$$

$$= \frac{1}{\Delta t}\left\{ \Delta t \int_{S(t)} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} + \int_{S(t+\Delta t)} \vec{B}(t)\cdot d\vec{a} - \int_{S(t)} \vec{B}(t)\cdot d\vec{a} \right\}$$

~~...~~ ②

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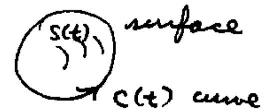
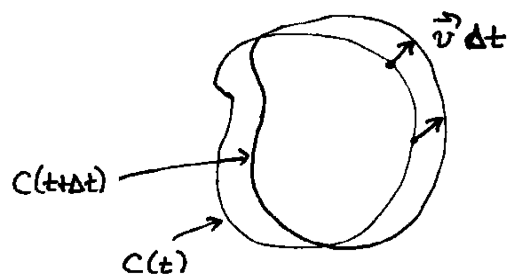
⑤ The case of a moving loop is important in practice (generators etc.). We still have

$$\oint_{C(t)} \vec{E}\cdot d\vec{\ell} = -\int_{S(t)} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

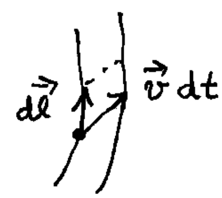


where integrals are taken at fixed time, but RHS is no longer $-\frac{dF}{dt}$ because an extra term arises due to $S(t)$. Let $\vec{v}$ = velocity of wire, a fn. of distance around wire. In time Δt, loop moves a little bit...



$$\frac{dF}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{ \int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot d\vec{a} - \int_{S(t)} \vec{B}(t)\cdot d\vec{a} \right\}$$

$$\{\ \} = \Delta t \int_{S(t)} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} + \underbrace{\int_{S(t+\Delta t)-S(t)} \vec{B}\cdot d\vec{a}}$$



on strip, $d\vec{a} = (\vec{v}\,dt)\times d\vec{\ell}$

$$\int_{\text{strip}} \vec{B}\cdot d\vec{a} = \Delta t \oint_{C(t)} \vec{B}\cdot(\vec{v}\times d\vec{\ell})$$

$$= -\Delta t \oint_{C(t)} (\vec{v}\times\vec{B})\cdot d\vec{\ell}$$

so $$\frac{dF}{dt} = \int_{S(t)} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} - \int_{C(t)} (\vec{v}\times\vec{B})\cdot d\vec{\ell}$$

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So...

$$\boxed{\oint_{C(t)} (\vec{E} + \vec{v}\times\vec{B})\cdot d\vec{l} = -\frac{dF}{dt}}$$ moving loop.

and now the EMF is *defined* to be the LHS, ~~this is also the~~ so Faraday's law (in words) EMF = − d/dt (flux) is still true, even for moving loops.

Physical interpretation of LHS = EMF. assuming a current I passing thru loop. → uniform, $\frac{\partial I}{\partial x} = 0$ Two points of view:

(a) In lab frame,

$$I \oint_{C(t)} \vec{E}\cdot d\vec{l} = \text{rate fields are doing work on currents in loop}$$

$$I \oint_{C(t)} \vec{v}\times\vec{B}\cdot d\vec{l} = \text{rate mechanical work being done on currents in loop}$$ (since someone has to move the loop against magnetic forces)

$$\text{EMF} = \left(\text{total rate work is being done on currents.}\right).$$

(b) In frame of loop (or small segment of loop):

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B} = \text{electric field in rest frame of loop (a Galilean transformation).}$$

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Now Magnetic Energy. Given some static $\vec{J}(\vec{x})$, produces some $\vec{B}(\vec{x})$. How much energy was required to set it up?

Have to step outside magnetostatics to answer, because if $\frac{\partial \vec{B}}{\partial t} \neq 0$ you don't have magnetostatics. But we want to build it up slowly, use quasistatic approximation: No radiation fields, retardation negligable over scale of system:

$$\frac{L}{c} = \text{light time across system} \ll \frac{1}{\omega_{typ}} = \text{time scale for changes in current.}$$

Will use:

$$\nabla \times \vec{H} = \vec{J}_f + \underbrace{\cancel{\frac{\partial \vec{D}}{\partial t}}}_{\text{neglect}}$$

$$\nabla \cdot \vec{J} + \underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{\text{neglect}} = 0$$

usual assumptions in magnetostatics, but actually we allow slow t-dep..

$$\nabla \times \vec{E} = -\underbrace{\frac{\partial \vec{B}}{\partial t}}_{\text{keep}}$$

Note, $\nabla \cdot \vec{J} = 0$ doesn't preclude t-dep of form $\vec{J} = \underbrace{\vec{J}_S(\vec{x})}_{\text{spatial part}} f(t)$ with $\nabla \cdot \vec{J}_S = 0$.

So current can rise and fall everywhere ~~at~~ in same proportion.

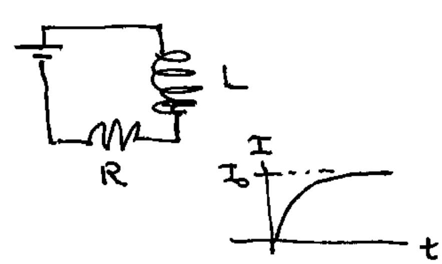
---

2 times: t $\quad$ $t+\delta t$

$\vec{B}(\vec{x})$ $\quad$ $\vec{B} + \delta\vec{B}$

$\vec{A}(\vec{x})$ $\quad$ $\vec{A} + \delta\vec{A}$

Ex:



If $\vec{B}$ is changing, by Faraday $\exists$ $\vec{E} \neq 0$, so battery must do work to keep charges (currents) moving against $\vec{E}$.

Want to compute $\delta W = \left(\text{work done by battery}\right)$ in time δt against $\vec{E}$.

$$\delta W = -\delta t \int_V \vec{E} \cdot \vec{J}\, dv.$$

(JDJ analyzes by breaking $\vec{J}$ up into current loops. But not general.)

(Sec. 6.6 Jackson.)

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Now turn to derivation of macroscopic Maxwell eqns, rigorous, w/o handwaving like up to now.

Assume validity of μ-scopic eqns...

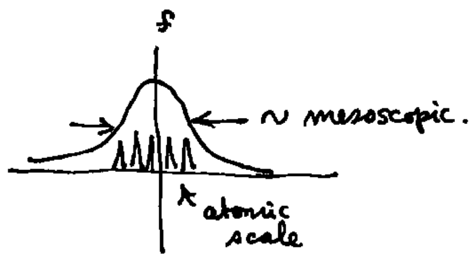
$\vec{e}, \vec{b} =$ μ-scopic fields.

$\eta =$ μ-scopic chg. density

$\vec{j} =$ μ-scopic current.

$$\boxed{\begin{aligned} \nabla\cdot\vec{e} &= \eta/\varepsilon_0 \\ \nabla\cdot\vec{b} &= 0 \\ \nabla\times\vec{e} &= -\frac{\partial\vec{b}}{\partial t} \\ \nabla\times\vec{b} &= \mu_0\vec{j} + \varepsilon_0\mu_0\frac{\partial\vec{e}}{\partial t} \end{aligned}}$$

Averaging accomplished by window function of mesoscopic size.. Gaussian convenient:



$$f(\vec{x}) = \frac{1}{(2\pi L^2)^{3/2}}\, e^{-r^2/2L^2}.$$

$L =$ mesoscopic length.

$\int d^3x\; f(\vec{x}) = 1$ normalized.

$f(\vec{x})$ rotationally symmetric, $f(-\vec{x}) = f(\vec{x})$.

Define averaged field:

$$\langle \vec{F}(\vec{x},t)\rangle \equiv \int d^3x' \quad F(\vec{x}',t)\; f(\vec{x}'-\vec{x})$$

$$= \int d^3x' \;\; F(\vec{x}+\vec{x}',t)\;\; f(\vec{x}') \qquad \text{shift origin}$$

$$= \int d^3\vec{x}' \;\; F(\vec{x}-\vec{x}',t)\;\; f(\vec{x}') \qquad \vec{x}' \to -\vec{x}'.$$

Henceforth suppress t-dep unless needed for clarity.

Then note, $\quad \nabla\langle F(\vec{x},t)\rangle = \int d^3\vec{x}' \;\nabla F(\vec{x}-\vec{x}')\, f(\vec{x}')$

$$= \langle \nabla F(\vec{x})\rangle.$$

Sim., $\quad \dfrac{\partial}{\partial t}\langle \vec{F}\rangle = \left\langle \dfrac{\partial F}{\partial t}\right\rangle.$