which integration by parts converts to

$$\delta W = \int dv \;\; \vec{E} \cdot \delta \vec{D}$$

That's about all you can say in general case.

also, will in general change volume which means work PdV is done etc.

But want to comment on aspect of this problem not discussed by Jackson, namely, when you bring in $\delta \rho_f$, you will change the temperature of the matter. In fact this whole discussion can't be divorced from thermodynamic state of the matter. What we have here is a generalized 1st law,

$$T dS = dU - \int dv (\delta \rho_f) \Phi \quad (+ P dV - X dY)$$

Jackson is mainly thinking of $T = 0$.

Ok, now if medium is linear, then $\vec{D} = \epsilon \vec{E}$ (think: $\epsilon = \epsilon(\vec{x})$ maybe)

$$\Rightarrow \quad \delta \vec{D} = \epsilon \, \delta \vec{E},$$

$$\delta W = \int dv \;\; \epsilon \;\; \vec{E} \cdot \delta \vec{E} = \delta \left(\frac{1}{2} \int \epsilon \, E^2 \, dv \right)$$

or $$W = \frac{1}{2} \int \epsilon E^2 dv = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, dv = \frac{1}{2} \int \rho_f \Phi \, dv.$$

Not done with electrostatics, but want to move on to magnetostatics (come back for some topics.)

Magnetic ~~phenomena~~ formalisms involve a lot of cross products, so now we make a next installment in tensor analysis... The Levi-Civita symbol.

Properties of Levi-Civita symbol.

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1. $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$

2. $(\vec{A}\times\vec{B})_i = \varepsilon_{ijk} A_j B_k$

$$(\nabla\times\vec{A})_i = \varepsilon_{ijk}\frac{\partial A_k}{\partial x_j} = \varepsilon_{ijk} A_{k,j}$$

Aside:

Comma notation for derivatives:

$$\frac{\partial f}{\partial x_i} = f_{,i}$$

$$\nabla\cdot\vec{A} = A_{i,i} \quad \text{etc.}$$

3. $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

$$\varepsilon_{ijk}\varepsilon_{ijl} = 2\,\delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} = \begin{vmatrix}\delta_{jl} & \delta_{jm}\\ \delta_{kl} & \delta_{km}\end{vmatrix}$$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix}$$

4. If $A_{ij} = -A_{ji}$, let $A_{ij} = \begin{pmatrix} 0 & A_z & -A_y\\ -A_z & 0 & A_x\\ A_y & -A_x & 0\end{pmatrix} = \varepsilon_{ijk} A_k$

Inverse, $A_i = \frac{1}{2}\varepsilon_{ijk} A_{jk}$.

5. $A_i B_j - A_j B_i = \varepsilon_{ijk}(\vec{A}\times\vec{B})_k$

6. M = 3×3 matrix.

$$\det M = \varepsilon_{ijk} M_{1i} M_{2j} M_{3k} = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn} M_{il} M_{jm} M_{kn}.$$

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Now magnetostatics. $\frac{\partial}{\partial t} = 0$ everywhere.

Continuity $\Rightarrow \nabla\cdot\vec{J} = 0$

<u>Maxwell</u> $\Rightarrow$ $\nabla\cdot\vec{B} = 0$ $\Rightarrow$ $\exists\ \vec{A}$ s.t. $\vec{B} = \nabla\times\vec{A}$.

$\nabla\times\vec{B} = \mu_0\vec{J}$

$\vec{A}$ subject to gauge fixing (not unique).

usu. use Coulomb gauge in magnetostatics, $\nabla\cdot\vec{A} = 0$ (Coulomb).

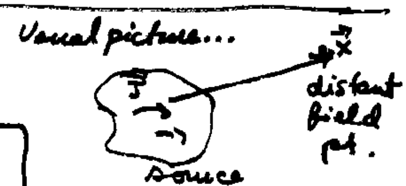
Then $\nabla\times(\nabla\times\vec{A}) = \nabla(\cancel{\nabla\cdot\vec{A}}) - \nabla^2\vec{A} = +\mu_0\vec{J}$

$$\nabla^2\vec{A} = -\mu_0\vec{J} \quad\Rightarrow\quad \boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d\tau'\ \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}.$$

c.f. $\nabla^2\Phi = -\rho/\varepsilon_0$

std. eqn, can use to find $\vec{A}$ of current loop etc. (For highly symmetrical problems, better to use Stokes' or Gauss' thm.)

For now look at ~~the~~ multipole expansion.



$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int dv' \ \vec{J}(\vec{x}')\left[\ \underset{\text{monopole}}{\frac{1}{r}} + \underset{\text{dipole}}{\vec{x}'\cdot\left(\frac{\vec{x}}{r^3}\right)} + \cdots\ \right]$$

Look at integrals that result over source: (drop primes.)

<u>monopole</u>: $\int dv\ \vec{J}(\vec{x}) = 0$ (it turns out). (if current distn is localized.)

<u>proof</u>: Consider

$$\frac{\partial}{\partial x_j}(x_i J_j) = \delta_{ij} J_j + x_i J_{j,j} \qquad J_{j,j} = \nabla\cdot\vec{J} = 0.$$

$$= J_i$$

Hence $\int_{vol} dv\ J_i = \int_{vol} dv\ \frac{\partial}{\partial x_j}(x_i J_j) = \int_{\partial} da\ x_i (n_j J_j)$

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or, $$\int_V \vec{J}\, dv = \int_S da\ \vec{x}(\hat{n}\cdot\vec{J})$$

Choose S outside region where ∃ current, get 0.

<u>No magnetic monopoles from ordinary electrical current.</u>

Can do a similar transformation on dipole term:

Want $\int dv\ x_i J_j$.

Consider $$\frac{\partial}{\partial x_k}(x_i x_j J_k) = \delta_{ik} x_j J_k + x_i \delta_{jk} J_k + \cancel{x_i x_j}(J_{k,k})$$
$$= x_i J_j + x_j J_i$$

when you integrate, get
$$\int (x_i J_j + x_j J_i)\, dv = 0.$$

$$\int x_i J_j\, dv = -\int x_j J_i\, dv = \frac{1}{2}\int dv\,(x_i J_j - x_j J_i)$$
$$= \frac{1}{2}\int dv\ \varepsilon_{ijk}(\vec{x}\times\vec{J})_k$$

so, dipole term is...

$$A_i = \frac{\mu_0}{4\pi}\frac{x_j}{r^3}\int dv'\ x'_j J_i(\vec{x}')$$

$$= \frac{\mu_0}{4\pi}\frac{x_j}{r^3}\varepsilon_{jik}\frac{1}{2}\int dv'\ \left[\vec{x}'\times\vec{J}(\vec{x}')\right]_k$$

$$= \frac{\mu_0}{4\pi}\vec{m}\times\left(\frac{\vec{x}}{r^3}\right) \quad \text{where} \quad \boxed{\vec{m} = \frac{1}{2}\int dv'\ \vec{x}'\times\vec{J}(\vec{x}')}$$

cf. $\vec{p} = \int dv'\ \vec{x}'\ \rho(\vec{x}')$

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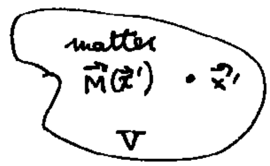
Just as with electric dipoles, so also with mag. dipoles we often adopt a continuum model.

$\cdot \vec{x}$ exterior field pt.



$$\vec{M} = \frac{\text{dipole mom.}}{\text{vol.}} = \underline{\text{magnetization}}$$

(Mention case of particle with intrinsic mag. moments.)

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int_V dv' \; \vec{M}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{\mu_0}{4\pi}\int_V dv' \; \frac{\vec{J}_b(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\mu_0}{4\pi}\int_S da' \; \frac{\vec{K}_b(\vec{x}')}{|\vec{x}-\vec{x}'|}.$$

Integration by parts.

<u>where</u> $\vec{J}_b = \nabla \times \vec{M}$

$\vec{K}_b = \vec{M} \times \hat{n}$

And, as in elec. case, $\vec{K}_b$ can be seen as special case of $\vec{J}_b$. (skip details)
Mention case when $\vec{x}$ is interior. note terminology b+f $\vec{B}, \vec{H}$
So look at Maxwell equ,

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \mu_0 (\vec{J}_b + \vec{J}_f) = \mu_0 \vec{J}_f + \mu_0 \nabla \times \vec{M}.$$

$$\boxed{\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}} \qquad \boxed{\nabla \times \vec{H} = \vec{J}_f} \quad \text{(magnetostatics only).}$$

General: $\vec{M} = \vec{M}[\vec{B}]$ (nonlinear).

Linear: $\vec{M} = \mu_0 \chi_m \vec{H}$ χ_m = magnetic susceptibility.

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1+\chi_m)\vec{H} = \mu \vec{H}$$

$\mu = (1+\chi_m)\mu_0$ = permeability