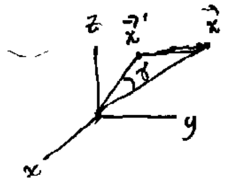
Fri. 9/5/02. ①

$r_< = \min(r, r')$
$r_> = \max(r, r')$

Summary:



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega') \qquad (3.70)$$

$$= \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma) \qquad (3.38)$$

where $\gamma = \angle(\vec{x}, \vec{x}')$, $\quad \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

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1. Derive Addn. Thm.

2. Intro to tensors, in prepn for multipole expansion.

For now...

A tensor is an object with indices on it. The rank is the # of indices.

A scalar has 0 indices	S	
" vector has 1 index	V_i	
" tensor of rank 2:	T_{ij}	
" " " " 3:	K_{ijk}	etc.

Define δ_{ij}, ϵ_{ijk}

We work in 3D space, so all indices take on values 1, 2, 3 or x, y, z.

Summation convention: Let $\vec{A} = M\cdot\vec{B}$ (vector = matrix · vector)

$$A_i = \sum_j M_{ij} B_j .$$

j index is repeated; it is a dummy index; it is summed.
i " " not repeated; it is free. Summ. conv. is to omit Σ on repeated indices. Thus:

$$A_i = M_{ij} B_j$$

$$\vec{A}\cdot\vec{B} = A_i B_i$$

$$\mathrm{tr}(M) = M_{ii}$$

Now irreducible tensors. Let B_{ij}, C_{ij} be rank 2 tensors.

Define scalar product, $\langle B, C\rangle \equiv B_{ij} C_{ij} = \mathrm{tr}(B^t C)$.

matrix product

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Note: Every antisymmetric tensor $\perp$ every symmetric.

$$\text{Let} \quad A_{ij} = -A_{ji}$$
$$S_{ij} = +S_{ji}$$

$$\langle A, S\rangle = A_{ij} S_{ij} = A_{ji} S_{ji} = (-A_{ij})(+S_{ij}) = -\text{self.} = 0.$$

In particular, every antisymm. tensor is $\perp$ identity:

$$\delta_{ij} A_{ij} = \langle I, A\rangle = A_{ii} = 0.$$

Some symmetric tensors are also $\perp$ to identity:

$$\langle I, S\rangle = \delta_{ij} S_{ij} = S_{ii} = \operatorname{tr} S.$$

So,

$$\operatorname{tr} S = 0 \quad \Leftrightarrow \quad \langle I, S\rangle = 0.$$

So, we see that space of rank 2, 3x3 tensors breaks up into 3 orthog. subspaces:

① Multiples of I: $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} = kI, \quad k\delta_{ij}$ 1

② Antisymmetric $A = \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{21} \\ -A_{13} & -A_{21} & 0 \end{pmatrix}$ 3

③ Symm. and traceless: $S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & -S_{11}-S_{22} \end{pmatrix}$ 5

# of indep. components

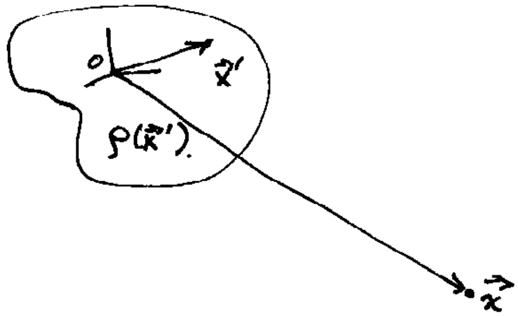
Note, $3 \times 3 = 9 = 1 + 3 + 5.$

cf. $1 \otimes 1 = 0 \oplus 1 \oplus 2$

So an arb. rank 2 tensor can be uniquely written as an l.c. of these.

$$\left.\begin{aligned} M_{ij} &= \underbrace{\tfrac{1}{2}(M_{ij}+M_{ji})} + \tfrac{1}{2}(M_{ij}-M_{ji}) = S_{ij} + A_{ij} \\ S_{ij} &= \left[S_{ij} - \tfrac{1}{3}\delta_{ij} \operatorname{tr}(S)\right] + \tfrac{1}{3}\delta_{ij}(\operatorname{tr} S) \end{aligned}\right\} \text{ irreducible subspaces}$$

3. Now multipole expansion in electrostatics. Localized charge distn, distant field pt.



Let $r = |\vec{x}|$
$r' = |\vec{x}'|$
$r' < r$

$$\Phi(\vec{x}') = \frac{1}{4\pi\varepsilon_0} \int dv' \, \rho(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|}$$

2 approaches: Cartesian, Spherical. Do Cartesian 1st.
Expand in $\vec{x}'$:

$$= \frac{1}{r} + \vec{x}' \cdot \frac{\vec{x}}{r^3} + \frac{1}{2} x_i' x_j' \left(\frac{3 x_i x_j - r^2 \delta_{ij}}{r^5} \right) + \cdots$$

monopole dipole quadrupole.

For monopole term get $\int dv' \rho(\vec{x}') = q$ = total charge = monopole moment.

accum. answers.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{q}{r} + \frac{\vec{p} \cdot \vec{x}}{r^3} + \frac{1}{2} Q_{ij} \frac{x_i x_j}{r^5} + \cdots \right\}$$

For dipole,
$$\vec{p} = \int dv' \, \vec{x}' \rho(\vec{x}')$$

For quadrupole: $\frac{3 x_i x_j - r^2 \delta_{ij}}{r^5}$ = symm. + traceless.

So can replace $x_i' x_j' \longrightarrow x_i' x_j' - \frac{1}{3} r'^2 \delta_{ij}$
Then can drop $\frac{-r^2 \delta_{ij}}{r^5}$.

So quadrupole term in expansion can be written:

$$\frac{1}{2}\left(3x_i' x_j' - \delta_{ij} r'^2\right) \cdot \frac{x_i x_j}{r^5}$$

So define $$\boxed{Q_{ij} = \int dv' \, (3x_i' x_j' - \delta_{ij} r'^2)\, \rho(\vec{x}').}$$ Symm. + traceless.

This is the Cartesian approach, illustrated thru $\ell=2$ term.

Now, spherical approach.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int dv' \rho(\vec{x}') \sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1} \cdot Y_{\ell m}(\Omega)\, Y^*_{\ell m}(\Omega').$$

define $$q_{\ell m} = \int dv' \, r'^{\ell}\, Y^*_{\ell m}(\Omega')\, \rho(\vec{x}')$$

$$\boxed{\Phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \sum_m \frac{q_{\ell m}\, Y_{\ell m}(\Omega)}{r^{\ell+1}}}$$

$q_{\ell m}$ = spherical components of multipole moments.

Leave as exercise to explore relation betw. sph., cartesian components.
Just note count: $\ell=0$ $n=1$ etc.
1 3
2 5

We know $\nabla^2\Phi = 0$ outside chg. distn, so Φ certainly has expansion,

$$\Phi(\vec{x}) = \sum_{\ell m} \frac{\Phi_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\Omega)$$ or $B_{\ell m}$

because that's what you get by separating the Laplace eqn. in sph. coords. in the exterior region. But question is, what are the coefs. $\Phi_{\ell m}$ in terms of chg. distn? Formalism above gives the answer.

bzh.