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__Problem:__ How to find G-fns. Only example so far used meth. of images (very special method).

First project today, get expansion of Green's fn in spherical coordinates. An example of an orthogonal expansion of Green's fns (there are several in Jackson). Do free-space G, simplest case. Already know answer:

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|}.$$

→ Results will be useful later.

__Idea:__ Solve

$$\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\,\delta(\vec{x}-\vec{x}\,')$$

in spherical coordinates.

Before we begin, remark that soln. of __Laplace__ eqn. in spherical coords is:

$$\nabla^2\Phi = 0$$

(just for orientation).

$$\Rightarrow \quad \Phi(r,\theta,\phi) = \sum_{\ell m}\left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}\right) Y_{\ell m}(\Omega).$$

__Write:__

for convenience

$$G(\vec{x},\vec{x}\,') = G(r\theta\phi;\, r'\theta'\phi') = \sum_{\substack{\ell m\\ \ell' m'}} G_{\ell m;\ell' m'}(r,r')\, Y_{\ell m}(\Omega)\, Y^*_{\ell' m'}(\Omega')$$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\nabla^2_{\Omega} \qquad \text{Use} \quad \nabla^2_{\Omega} Y_{\ell m} = -\ell(\ell+1)\, Y_{\ell m}$$

$$\nabla^2 G = \sum_{\substack{\ell m\\ \ell' m'}} \underbrace{\left[\frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\ell(\ell+1)}{r^2}\right]}_{\text{Call this } \mathcal{D}\text{, acts on } r \text{ only.}} G_{\ell m;\ell' m'}(r,r')\, Y_{\ell m}(\Omega)\, Y^*_{\ell' m'}(\Omega')$$

$$= -4\pi\, \frac{\delta(r-r')\,\delta(\theta-\theta')(\phi-\phi')}{r^2 \sin\theta}.$$

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Now mult by $\int d\Omega\, d\Omega'\, Y^*_{LM}(\Omega)\, Y_{L'M'}(\Omega')$,

use orthonormality of $Y_{\ell m}$'s. on LHS, picks out single term.

$$\boxed{\mathcal{D}\, G_{\ell m;\ell' m'}(r,r') = -4\pi \frac{\delta(r-r')}{r^2}\, \delta_{\ell\ell'}\delta_{mm'}}$$

on RHS, can do integral because of δ-fns,

$$-4\pi \frac{\delta(r-r')}{r^2} \underbrace{\int d\Omega\, d\Omega' \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta} Y^*_{\ell m}(\Omega)\, Y_{L'M'}(\Omega')}_{\int d\Omega\, Y^*_{LM}(\Omega) Y_{L'M'}(\Omega) = \delta_{LL'}\delta_{MM'}.}$$

In case $\ell \neq \ell'$ or $m \neq m'$, RHS $=0$:

$$\mathcal{D}\, G_{\ell m;\ell' m'}(r,r') = 0. \quad \Rightarrow \quad G_{\ell m;\ell' m'} = A r^{\ell} + \frac{B}{r^{\ell+1}}.$$

Both $A = B = 0$ since working in free space (finite at $r \to 0$, zero at $r \to \infty$).

So, $G_{\ell m;\ell' m'}(r,r') = \delta_{\ell\ell'}\delta_{mm'}\, g_{\ell m}(r,r')$ where

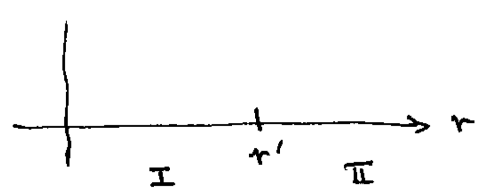
$$\mathcal{D}\, g_{\ell m}(r,r') = -4\pi \frac{\delta(r-r')}{r^2}.$$

$$= \left[\frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\ell(\ell+1)}{r^2}\right] g_{\ell m}(r,r')$$

How to solve. RHS $=0$ except at $r=r'$. So have 2 regions,

I. $r<r'$

II. $r>r'$



$$g_{I}(r,r') = A r^{l} \qquad \text{(finite at } r=0\text{)}$$

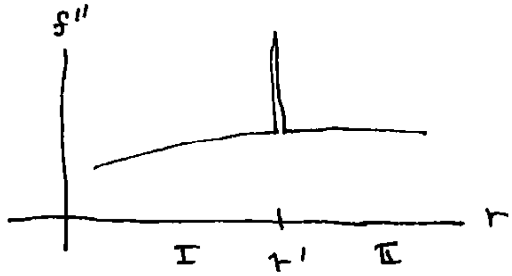
$$g_{II}(r,r') = \frac{B}{r^{l+1}} \qquad (\to 0 \text{ at } r\to\infty).$$

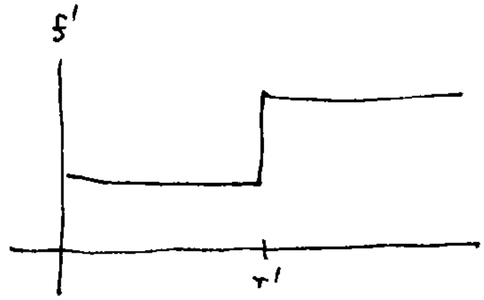
Just need to match at $r=r'$. ~~suppress lm subscript for ease.~~

Write equ, $f = r g_{em}$.

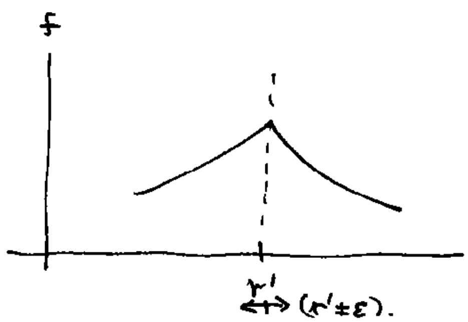
$$\frac{d^2 f}{dr^2} - \frac{l(l+1)}{r^2} f = -4\pi \frac{\delta(r-r')}{r}.$$

f'' must have δ-fn singularity. to balance RHS.





f' is discontinuous



f is contin. but
f has disc. in slope

So integrate ode, $\displaystyle\int_{r'-\varepsilon}^{r'+\varepsilon} dr \; : \; \left(f'_{II} - f'_{I}\right)\Big|_{r=r'} + 0 = -\frac{4\pi}{r'}.$

$f_I = A r^{\ell+1}$ $\qquad f_I' = (\ell+1) A r^\ell$

$f_{II} = \frac{B}{r^\ell}$ $\qquad f_{II}' = -\frac{\ell B}{r^{\ell+1}}$.

2 eqns:

1. Continuity of f at $r = r'$: $\qquad A r'^\ell = \frac{B}{r'^{\ell+1}}$

2. Prescribed discontinuity in f'' at $r = r'$: $\qquad \frac{-\ell B}{r'^{\ell+1}} - (\ell+1) A r'^\ell = -\frac{4\pi}{r'}$

Solve for A, B, find $\qquad A = \frac{4\pi}{2\ell+1} \frac{1}{r'^{\ell+1}}$

$$B = \frac{4\pi}{2\ell+1} r'^{\ell}$$

$$g_{\ell m}(r,r') = \frac{4\pi}{2\ell+1} \begin{Bmatrix} \frac{r^\ell}{r'^{\ell+1}}, & r < r' \\ \frac{r'^\ell}{r^{\ell+1}}, & r > r' \end{Bmatrix} = \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \qquad \begin{matrix} r_< = \min(r,r') \\ r_> = \max(r,r') \end{matrix}$$

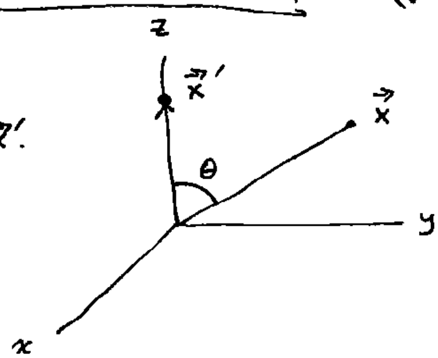
Now put it back together, you get

$$\boxed{\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega')}$$

very useful result.

(J. 3.70)

Now special case, put $\vec{x}'$ on z-axis.
Then coord θ of $\vec{x}$ is also angle betw. $\vec{x}, \vec{x}'$.



Also, $\theta' = 0$, $\cos\theta' = 1$.

$$Y_{\ell m}(\Omega') = \sqrt{\frac{2\ell+1}{4\pi}}\, \delta_{m0}$$

so m-sum reduces to single $\overset{m=0}{\text{term}}$, and $\quad Y_{\ell 0}(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$.

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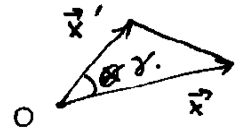
<u>Result:</u> $\vec{x}'$ on z-axis: $(\theta' = 0)$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta).$$

Now argue geometrically. $O, \vec{x}, \vec{x}'$ form triangle.



LHS is rotational invariant of Δ. So RHS must be too. Therefore it is indep. of coordinate system, as long as we interpret θ, <u>not</u> as coord. of $\vec{x}$, but as angle betw $\vec{x}, \vec{x}'$. To avoid confusion, introduce new symbol, γ

$$\gamma = \angle(\vec{x}, \vec{x}'), \quad \frac{\vec{x}\cdot\vec{x}'}{rr'} = \cos\gamma$$

Then
$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)}. \qquad (J.\ 3.38)$$

Now restore coordinates, but <u>don't</u> place $\vec{x}'$ on z-axis: Then

$$\cos\gamma = \hat{r}\cdot\hat{r}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \qquad \hat{r} = \frac{\vec{x}}{r}, \quad \hat{r}' = \frac{\vec{x}'}{r'}$$

bzh ~~bzh~~.

Finally, compare expansion above with this one, get

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_{lm}(\Omega)\, Y_{lm}^*(\Omega')} \qquad (J.\ 3.62).$$

<u>Addition thm. for sph. harmonics.</u>