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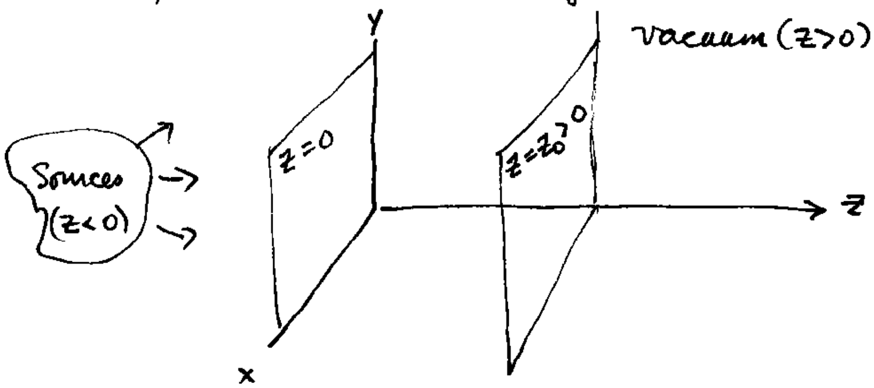
Intro to <u>diffraction</u> today. Let there be sources of light in region $z<0$, but vacuum in region $z>0$.



Let $\psi(\vec{x},t)$ stand for E_x, E_y or E_z. It satisfies the wave eqn,

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

in region $z>0$; all eqns below refer only to $z>0$. We suppose we don't know much about sources in region $z<0$ but we do have knowledge of ψ on plane $z=0$. Given this knowledge, we wish to find ψ on other planes $z=z_0>0$, effectively we wish to "propagate" ψ in z.

The medium is static so we F.T. in time to get $\psi(\vec{x},\omega)$. Henceforth we work at some fixed ω. Thus ψ satisfies the Helmholtz eqn,

$$\nabla^2\psi + k_0^2\psi = 0 \quad , \qquad \psi = \psi(\vec{x},\omega)$$

where $k_0 \equiv \frac{\omega}{c}$ = fixed wave number for the rest of this discussion.

Coordinate z will be treated differently from other coordinates. Thus we write $\psi(\vec{x}) = \psi(\vec{x}_\perp, z)$ where $\vec{x}_\perp = (x,y)$, coordinates on transverse plane. Henceforth suppress ω dependence. Write Helmholtz eqn as

$$-\frac{\partial^2\psi}{\partial z^2} = (k_0^2 + \nabla_\perp^2)\psi(\vec{x}_\perp, z).$$

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Introduce notation $\hat{\vec{k}}_\perp = -i\nabla_\perp$ where the $\wedge$ means "operator". Something without a hat is a number or vector of numbers, e.g. $\vec{k}_\perp = (k_x, k_y)$ means a wave vector in ordinary sense. Notice $\hat{k}_\perp^2 = -\nabla_\perp^2$, so

$$-\frac{\partial^2\psi}{\partial z^2} = (k_0^2 - \hat{k}_\perp^2)\psi.$$

To solve introduce Fourier transforms in $\vec{x}_\perp$, the usual def'n:

$$\tilde{\psi}(\vec{k}_\perp, z) = \int \frac{d^2\vec{x}_\perp}{2\pi}\, e^{-i\vec{k}_\perp\cdot\vec{x}_\perp}\, \psi(\vec{x}_\perp, z)$$

($\sim$ Here means Fourier transform)

$$\psi(\vec{x}_\perp, z) = \int \frac{d^2\vec{k}_\perp}{2\pi}\, e^{+i\vec{k}_\perp\cdot\vec{x}_\perp}\, \tilde{\psi}(\vec{k}_\perp, z).$$

Then the Helmholtz eqn is

no hat now, just a number.

$$-\frac{\partial^2\tilde{\psi}(\vec{k}_\perp, z)}{\partial z^2} = (k_0^2 - k_\perp^2)\,\tilde{\psi}(\vec{k}_\perp, z)$$

The general solution is

$$\tilde{\psi}(\vec{k}_\perp, z) = \begin{cases} A e^{ik_z z} + B e^{-ik_z z} & k_0^2 > k_\perp^2 \\ A e^{-\kappa_z z} + B e^{+\kappa_z z} & k_0^2 < k_\perp^2 \end{cases}$$

where

$$\left.\begin{aligned} k_z &= \sqrt{k_0^2 - k_\perp^2} \\ \kappa_z &= \sqrt{k_\perp^2 - k_0^2} \end{aligned}\right\},$$

k_z, κ_z are understood as fns of $\vec{k}_\perp$.

But we reject the B solutions on physical grounds, since all sources are in region $z<0$ and we cannot have any waves that are propagating in the $-z$ direction nor any waves that blow up as $z \to \infty$.

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As for the A solutions, they satisfy a simpler eqn than the Helmholtz eqn,

$$i \frac{\partial \tilde{\psi}}{\partial z}(\vec{k}_\perp, z) = -\sqrt{k_0^2 - k_\perp^2}\ \tilde{\psi}(\vec{k}_\perp, z)$$

where the $\sqrt{\ }$ is interpreted as

$$\sqrt{k_0^2 - k_\perp^2} = \begin{cases} \text{usual } + \sqrt{\ } & \text{if } k_\perp < k_0 \\ i\sqrt{k_\perp^2 - k_0^2} & \text{if } k_\perp > k_0. \end{cases}$$

This is a pseudo-Schrödinger eqn, every solution of it is also a soln of the Helmholtz eqn, but the converse is not true since the Helmholtz eqn has solns we are not interested in physically.

If the pseudo-Schrödinger eqn is F.T.'ed back to $\vec{x}_\perp$ space it becomes

$$i \frac{\partial \psi(\vec{x}_\perp, z)}{\partial z} = -\sqrt{k_0^2 - \hat{k}_\perp^2}\ \psi(\vec{x}_\perp, z)$$

$$= -\sqrt{k_0^2 + \nabla_\perp^2}\ \psi(\vec{x}_\perp, z)$$

where the $\sqrt{\ }$ involving the Laplacian is interpreted by going to $\vec{k}_\perp$-space where $\hat{\vec{k}}_\perp$ is purely multiplicative and the $\sqrt{\ }$ is interpreted as above. We also write this as

$$i \frac{\partial \psi}{\partial z} = \hat{H}\psi \qquad \text{where} \quad \hat{H} = -\sqrt{k_0^2 - \hat{k}_\perp^2}$$

to make it look like the Schr. eqn.

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This pseudo-Schrödinger eqn has the solution,

$$\psi(\vec{x}_\perp, z) = e^{-i\hat{H}z}\, \psi(\vec{x}_\perp, 0)$$

or

$$\psi(\vec{x}_\perp, z) = \int d^2\vec{x}_\perp{}' \; K(\vec{x}_\perp, \vec{x}_\perp{}', z)\; \psi(\vec{x}_\perp{}', 0)$$

where $K(\vec{x}_\perp, \vec{x}_\perp{}', z)$ is the kernel of the operator $e^{-i\hat{H}z}$, i.e.

$$K(\vec{x}_\perp, \vec{x}_\perp{}', z) = \langle \vec{x}_\perp | \, e^{-i\hat{H}z} \, | \vec{x}_\perp{}' \rangle$$

in Dirac notation. This kernel can be worked out by going to $\vec{k}_\perp$-space where it is trivial. We won't need the result, but for reference it is

$$K(\vec{x}_\perp, \vec{x}_\perp{}', z) = \frac{e^{ik_0 R}}{2\pi}\, z \left(\frac{1}{R^3} - \frac{ik_0}{R^2} \right)$$

where

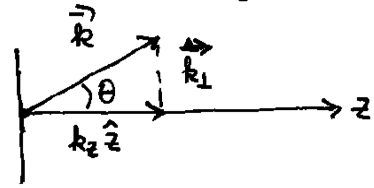
$$R = \sqrt{(\vec{x}_\perp - \vec{x}_\perp{}')^2 + z^2} = \text{dist. from "source" to "field" pt.}$$

Instead we will make an approximation that simplifies K. We will assume that

$$k_\perp \ll k_0$$

Called the <u>paraxial approximation</u>, it means that the wave vectors $\vec{k}$ which contribute to the wave field in region $z > 0$ are mostly pointing in the z-direction, i.e.

θ = small



$|\vec{k}| = k_0$.

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In the paraxial approx, we can expand the $\sqrt{\ }$ in $\hat{H}$,

$$\hat{H} = -\sqrt{k_0^2 - \hat{k}_\perp^2} \approx -k_0 + \frac{\hat{k}_\perp^2}{2k_0}.$$

Thus

$$\langle \vec{x}_\perp | e^{-i\hat{H}z} | \vec{x}_\perp' \rangle = e^{ik_0 z} \langle \vec{x}_\perp | e^{-i\frac{\hat{k}_\perp^2}{2k_0}z} | \vec{x}_\perp' \rangle .$$

To evaluate the final matrix element, we recall that in QM, if

$$H = \frac{p^2}{2m}$$

then $\langle \vec{x} | e^{-iHt/\hbar} | \vec{x}' \rangle = \left(\frac{m}{2\pi i\hbar t}\right)^{n/2} e^{\frac{im}{2\hbar t}(\vec{x}\cdot\vec{x}')^2}$

where n = # of dimensions. So we just change notation,

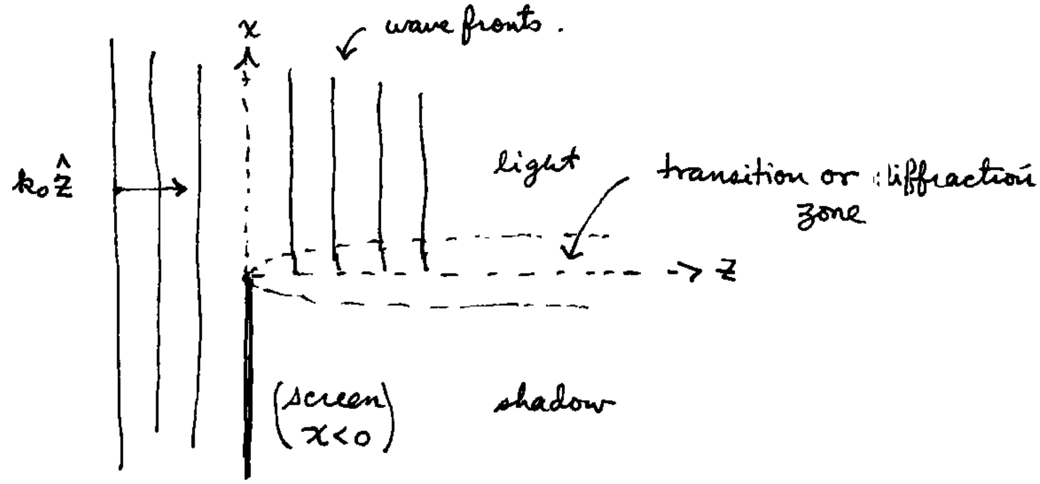
$$\hbar = 1$$
$$n = 2$$
$$m \to k_0$$
$$t \to z$$
$$\vec{x}, \vec{x}' \to \vec{x}_\perp, \vec{x}_\perp',$$

and we have our propagator in the paraxial approx,

$$K(\vec{x}_\perp, \vec{x}_\perp', z) = e^{ik_0 z} \left(\frac{k_0}{2\pi i z}\right) e^{\frac{ik_0}{2z}(\vec{x}_\perp - \vec{x}_\perp')^2}$$

You can also get this from the exact K above by expanding r, and assuming $z \gg |\vec{x}_\perp - \vec{x}_\perp'|$. You also need to assume $z \gg \lambda = \frac{k_0}{2\pi}$.

Now consider diffraction from the ½ plane. Let region of plane $z=0$ be occupied by a screen, and suppose a plane wave with $\vec{k} = k_0\hat{z}$ is incident from the left (for simplicity). Fr. 12/6/02.



The waves are cut off by the screen. according to geometrical optics, the plane $x=0$ is the abrupt transition between light and shadow, but actually there is a transition or diffraction zone in which the wave field continuously decays to zero. So how big is the diffraction zone, and what does the wave field look like in that zone?

Use our propagator,

$$\psi(\vec{x}_\perp, z) = e^{ik_0 z}\left(\frac{k_0}{2\pi i z}\right)\int d^2\vec{x}_\perp' \; e^{\frac{ik_0}{2z}(\vec{x}_\perp-\vec{x}_\perp')}\,\psi(\vec{x}_\perp', 0).$$

But what do we use for $\psi(\vec{x}_\perp, 0)$? We use the <u>Kirchhoff approximation</u>, which says that the wave field inside the aperture of a diffracting system is the same as what it would be without the screen. This is incorrect near the edge of the screen (a few wavelengths) but if the dimensions of the aperture are large compared to λ it will usually be a good approx. For this problem, the aperture is ∞ in size (region $x>0$).

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This means for this problem we set

$$\psi(\vec{x}_\perp,0) = \psi(x,y,0) = \begin{cases} \psi_0, \text{ a const}, & x>0 \\ 0, & x<0. \end{cases}$$

So the integral becomes,

$$\psi(x,y,z) = e^{ik_0 z}\left(\frac{k_0}{2\pi i z}\right)\int_0^\infty dx' \int_{-\infty}^{+\infty} dy' \; e^{\frac{ik_0}{2z}\left[(x-x')^2+(y-y')^2\right]} \; \psi_0$$

y' integral is easy, $$\int_{-\infty}^{+\infty} dy' \; e^{i\frac{k_0}{2z}(y-y')^2} = \sqrt{\frac{2\pi i z}{k_0}}$$

so

$$\psi(x,y,z) = e^{ik_0 z}\sqrt{\frac{k_0}{2\pi i z}}\;\psi_0 \int_0^\infty dx' \; e^{\frac{ik_0}{2z}(x'-x)^2}$$

Remaining integral can't be done in terms of elementary fns, but we can put it in a standard form by writing,

$$s = \sqrt{\frac{k_0}{2z}}\,(x'-x),$$

so

$$\psi(x,y,z) = \psi_0 \; F\left(-x\sqrt{\frac{k_0}{2z}}\right),$$

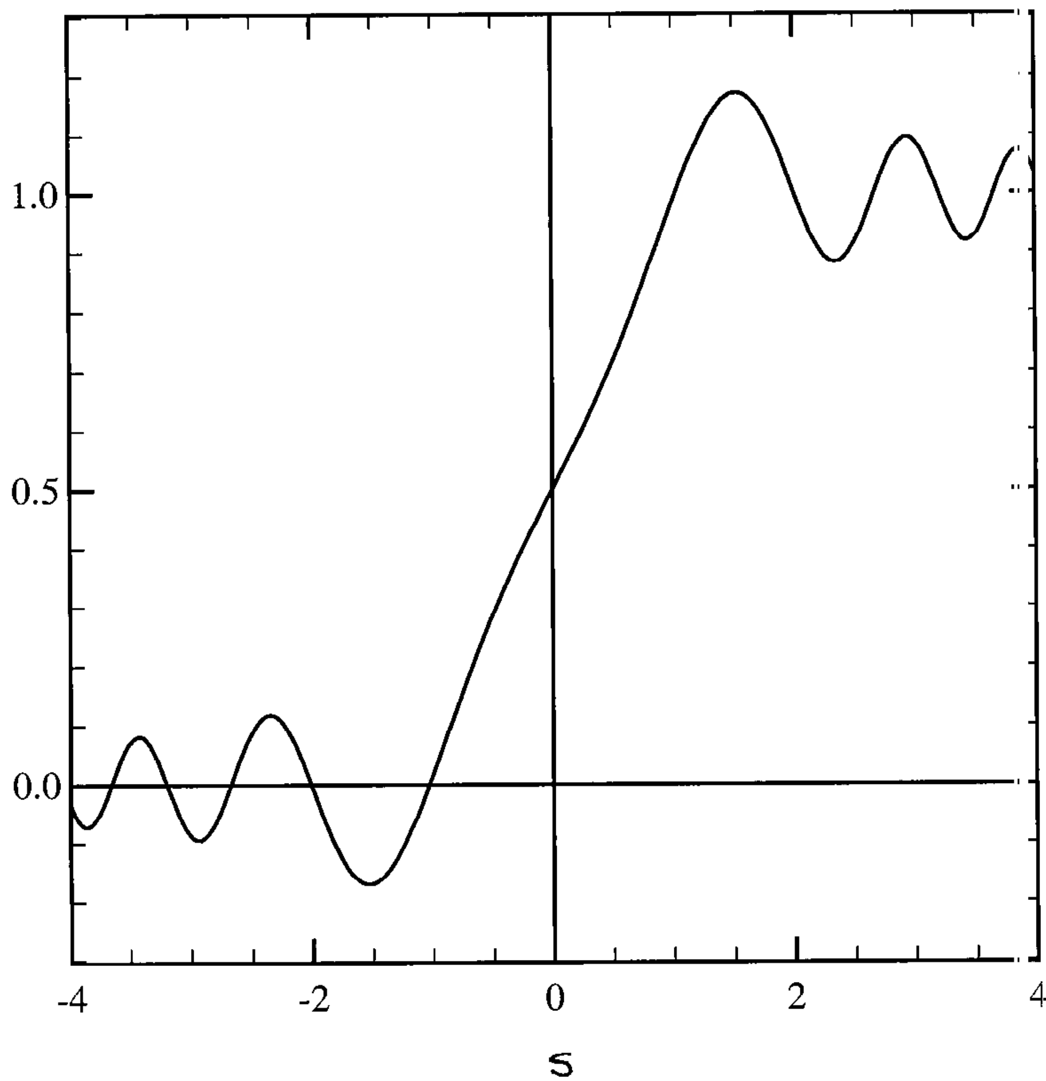
where $$F(s) = \frac{1}{\sqrt{i\pi}}\int_s^\infty ds' \; e^{is'^2}.$$

$F(s)$ is a <u>Fresnel integral</u> and it has properties,

$$F(-\infty) = 1 \qquad F(0) = 1/2 \qquad F(\infty) = 0$$

see plot.

Fresnel Integral $F(-s)$



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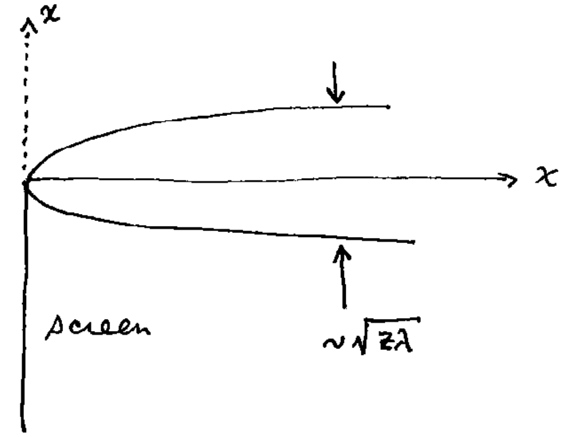
So the transition from light to shadow has the same functional form (as a function of x), independent of z, but the scale length (in x), that is the width of the transition or diffraction zone, depends on z. In fact, the transition takes place over an interval $\Delta s = O(1)$, therefore $\Delta x \sim \sqrt{\frac{2z}{k_0}} \sim \sqrt{z\lambda}$.

The diffraction zone has a parabolic shape, whose width is $\sim \sqrt{z\lambda}$.



This is for an semi-infinite aperture. As z increases, the diffraction zone grows and eats into both the light and shadow regions.

Something similar happens with apertures of other shapes. If the aperture is finite in size, then if z gets big enough the diffraction zone grows and consumes the entire light zone. Let a = dimensions of the aperture. Then if

$$\sqrt{z\lambda} \ll a$$

we have distinguishable light zone, surrounded by small diffraction zone before we go into shadow. If $\sqrt{z\lambda} \gg a$, then we have only diffraction zone. Names given:

$\sqrt{z\lambda} \ll a$	Fresnel diffraction
$\sqrt{z\lambda} \gg a$	Fraunhofer diffraction