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Summary.

$$\vec{P}(\vec{x},t) = \epsilon_0 \int_{-\infty}^{t} dt' \, \chi_e(t-t') \, \vec{E}(\vec{x},t') = \epsilon_0 \int_0^{\infty} d\tau \, \chi_e(\tau) \, \vec{E}(\vec{x}, t-\tau)$$

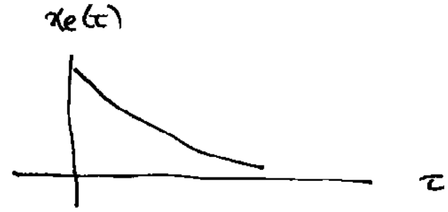
$$\vec{P}(\vec{x},\omega) = \epsilon_0 \chi_e(\omega) \, \vec{E}(\vec{x},\omega)$$

$\chi_e(\tau) = 0$ for $\tau < 0$ (causality)

$$\chi_e(\omega) = \int_0^{\infty} d\tau \, e^{i\omega\tau} \chi_e(\tau) .$$

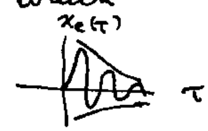
Dielectric only for now (not conductor)

Now $\chi_e(\tau)$ (or $G(\tau)$) falls off with τ, on a time scale which is the relaxation time of the polarization. This means that, although $\vec{P}(t)$ is affected by $\vec{E}(t')$ at previous times, it does not go too far back in time. Thus $\chi_e(\tau)$ must die off somehow with increasing τ,



In fact, Jackson studies a single resonance model, for which

$$\chi_e(\tau) \sim \sin \omega_1 \tau \, e^{-\gamma\tau/2}$$



where γ is the damping of the resonance. So actually it oscillates as it dies off.

This means that $\chi_e(\omega)$ (the Fourier transform) is defined for all real ω. In fact it is defined for complex ω in the upper half plane.

If we write $\omega = \omega' + i\omega''$ for Re, Im parts of ω, then

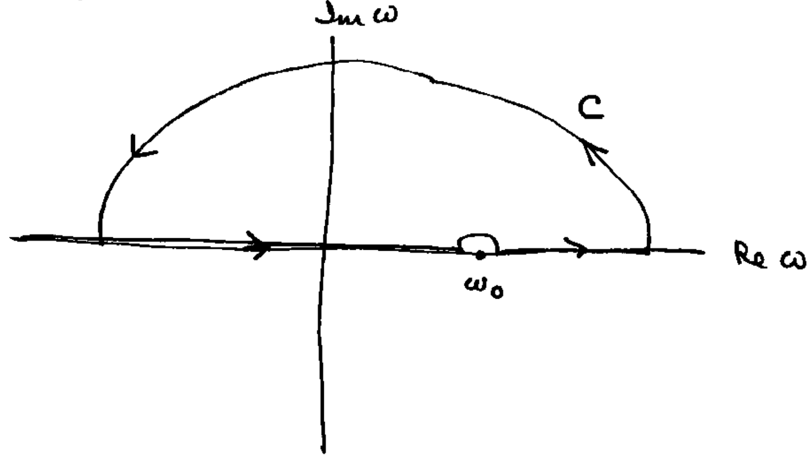
$$\chi_e(\omega) = \int_0^\infty d\tau \; e^{i\omega'\tau} \, e^{-\omega''\tau} \, \chi_e(\tau),$$

so $\chi_e(\omega)$ is defined for any $\text{Im}\,\omega = \omega'' > 0$. In fact, $\chi_e(\omega)$ and all of its derivatives wrt ω exist for $\text{Im}\,\omega > 0$: $\chi_e(\omega)$ is an <u>analytic</u> fn. of ω in upper ½ plane. Also, because of the exponential damping, $\chi_e(\omega) \to 0$ as $\omega \to \infty$ in the upper ½ plane.

(Taking ω derivs ~~pos~~ brings down powers of τ, but these are dominated by exponential decay in $e^{-\omega''\tau}$, and in $\chi_e(\tau)$ itself).

Emphasize that analyticity of $\chi_e(\omega)$ in upper ½ plane depends on $\chi_e(\tau) = 0$ for $\tau < 0$ (otherwise integral would have diverging exponential) which in turn is a consequence of <u>causality</u>.

so consider complex ω plane, pick a real value ω_0 and consider a Cauchy-like integral along contour C:



This is for a dielectric. For a conductor $\chi_e(\omega)$ has a pole at $\omega=0$ (since it contains the term $i\sigma/\omega$) and we must bump around $\omega=0$ too. (In addition to $\omega=\omega_0$). Now consider

$$\int_C d\omega \; \frac{\chi_e(\omega)}{\omega-\omega_0} .$$

The integrand is analytic inside C, so the integral vanishes. But $\chi_e(\omega)\to 0$ as $\omega\to\infty$ in the upper ½ plane, so the integral over the large semicircle vanishes (in the limit the radius $\to\infty$). The integral over the small semicircle is (in limit radius $\to 0$)

$$\int_{\substack{\text{small}\\ \text{semicircle}}} d\omega \; \frac{\chi_e(\omega)}{\omega-\omega_0} = -i\pi\, \chi_e(\omega_0)$$

The rest of the integral (along the real axis) is defined as the "principal part",

$$\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{\omega_0-\varepsilon} + \int_{\omega_0+\varepsilon}^{+\infty}\right) d\omega \; \frac{\chi_e}{\omega-\omega_0} \equiv P\int_{-\infty}^{+\infty} d\omega \; \frac{\chi_e}{\omega-\omega_0} .$$

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Result is,

$$i\pi \chi_e(\omega) = \mathcal{P}\int_{-\infty}^{+\infty} d\omega' \; \frac{\chi_e(\omega')}{\omega' - \omega}$$

where we have changed notation, $\omega \to \omega'$, $\omega_0 \to \omega$.

Now take Re, Im parts of this,

$$\left.\begin{aligned} \operatorname{Im} \chi_e(\omega) &= \frac{-1}{\pi} \mathcal{P}\int_{-\infty}^{+\infty} d\omega' \; \frac{\operatorname{Re} \chi_e(\omega')}{\omega' - \omega} \\ \operatorname{Re} \chi_e(\omega) &= \frac{1}{\pi} \mathcal{P}\int_{-\infty}^{+\infty} d\omega' \; \frac{\operatorname{Im} \chi_e(\omega')}{\omega' - \omega} \end{aligned}\right\} \text{Kramers-Kronig relations.}$$

K-K relations show that if you have a real part, you must also have an imaginary part: dispersion $\Leftrightarrow$ dissipation.

K-K relations usually written in a different form. Note that

$$\chi_e(\omega) = \chi_e(-\omega)^* \qquad \text{when } \omega = \text{real, since } \chi_e(\tau) = \text{real.}$$

$$\Rightarrow \quad \operatorname{Re} \chi_e(\omega) + i \operatorname{Im} \chi_e(\omega) = \operatorname{Re} \chi_e(-\omega) - i \operatorname{Im} \chi_e(-\omega)$$

$$\Rightarrow \quad \begin{aligned} \operatorname{Re} \chi_e(\omega) &= \text{even in } \omega \\ \operatorname{Im} \chi_e(\omega) &= \text{odd in } \omega. \end{aligned}$$

So,

$$\begin{aligned} \operatorname{Im} \chi_e(\omega) &= \tfrac{1}{2}\left[\operatorname{Im} \chi_e(\omega) - \operatorname{Im} \chi_e(-\omega) \right] \\ &= -\frac{1}{2\pi} \mathcal{P}\int_{-\infty}^{+\infty} d\omega' \; \operatorname{Re} \chi_e(\omega') \left(\frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega} \right) \\ &= -\frac{\omega}{\pi} \mathcal{P}\int_{-\infty}^{+\infty} d\omega' \; \frac{\operatorname{Re} \chi_e(\omega')}{\omega'^2 - \omega^2} \end{aligned}$$

$$\boxed{\operatorname{Im} \chi_e(\omega) = -\frac{2\omega}{\pi} \mathcal{P}\int_{0}^{\infty} d\omega' \; \frac{\operatorname{Re} \chi_e(\omega')}{\omega'^2 - \omega^2}}$$

Similarly, you find

$$\boxed{\mathrm{Re}\, \chi_e(\omega) = \frac{2}{\pi} \, P \int_0^{+\infty} \omega' d\omega' \; \frac{\mathrm{Im}\, \chi_e(\omega')}{\omega'^2 - \omega^2}}$$

The 2nd of these is especially useful, e.g. experimental data on the absorption of radiation can be used to compute the real part of the susceptibility.

The KK relations can be used in various ways. For example, if $\omega \to \infty$ then the ω^2 dominates in the denom. of the integral for $\mathrm{Re}\, \chi_e(\omega)$,

$$\omega \to \infty, \quad \chi_e(\omega) \approx -\frac{1}{\omega^2} \; \frac{2}{\pi} \int_0^{\infty} \omega' d\omega' \; \mathrm{Im}\, \chi_e(\omega').$$

But we know that $\chi_e(\omega) \to -\frac{\omega_p^2}{\omega^2}$ so

$$\omega_p^2 = \frac{2}{\pi} \int_0^{\infty} \omega' d\omega' \; \mathrm{Im}\, \chi_e(\omega')$$

Jackson uses KK relations to derive another interesting result,

$$\int_0^{\infty} d\omega \; \mathrm{Re}\, \chi_e(\omega) = 0.$$

The average of χ_e over all ω vanishes; the avg. of ϵ is 1. But the derivation of this is somewhat tricky, will skip it.