Mon 12/2/02

Now look at high ω limit. If $\omega \gg$ any ω_j, then

$$\epsilon = \epsilon_0 \left\{ 1 - \frac{\omega_p^2}{\omega^2} \right\}.$$

This is the dielectric fn. of an (cold) electron plasma, hence the name ω_p (plasma freq.). Notice that it gives $\epsilon(\omega) < 0$ if $\omega < \omega_p$. Since

$$k^2 = \omega^2 \epsilon(\omega) \mu_0 ,$$

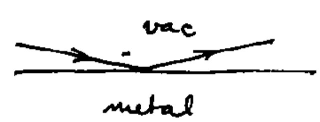
if $\epsilon < 0$, k is purely imaginary and waves damp. For this reason, waves with $\omega < \omega_p$ are reflected from a plasma (this is why radio waves bounce off the ionosphere.)

In the case of metals, something like this happens at optical frequencies. (That is, ϵ goes negative.) Notice that each $\omega_j <$ some given ω contributes a negative term to ϵ; in the case of metals, must include the contribution from conduction electrons, which for $\omega \gg \gamma_0$ give

$$- \frac{f_0}{Z} \frac{\omega_p^2}{\omega^2}$$

contribution to ϵ. This helps make ϵ go negative. This is why metals reflect light so well. But when $\omega > \omega_p$, ϵ becomes > 0 and metals become transparent. This happens in UV.

at higher ω (X-ray) can use metals with glancing reflection to make mirrors, for X-ray optics:



this works by total internal reflection, if the angle is small enough, because $\epsilon_{metal} = 1 - \frac{\omega_p^2}{\omega^2} < 1 = \epsilon_{vac}$.

Mon 12/2/02.

Now model of plasma. The limiting form ($\omega \gg \omega_j$) for ϵ,

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

only applies at high frequencies ($\omega \gg \omega_p$) for conductors and dielectrics (solids), but is valid over much wider range of ω for plasmas, in particular, even when $\omega < \omega_p$. So now look at the plama model.

Plasma model:

1. Light electrons, heavy ions (ignore ion dynamics)
2. Smear ions into uniform background with uniform $\rho_0 > 0$.
3. Cold electron fluid model.

Electron fluid equs:

n = # electrons/vol (= NZ when comparing to resonance model of solids)

$\rho = -ne$ = charge density

$\rho_m = mn$ = mass density (m = elec. mass).

$\vec{u}$ = fluid velocity.

$$\frac{\partial \rho_m}{\partial t} + \nabla\cdot(\rho_m \vec{u}) = 0 \qquad \text{(mass conservation)}$$

$$\rho_m\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = \rho\,(\vec{E} + \vec{u}\times\vec{B}) \qquad \text{(force equ)}$$

$$\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho + \rho_0) \qquad \text{(Gauss' law)}$$

+ 3 other Maxwell equs.

Mon 12/2/02.

Step 1, use $\rho = -ne$, $\rho_m = nm$ to eliminate ρ, ρ_m:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0$$

$$m\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -e\left(\vec{E} + \vec{u}\times\vec{B}\right)$$

$$\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\left(-en + \rho_0\right)$$

These are nonlinear equations. Step 2 is to linearize them about an equilibrium solution. The equilibrium solution is a uniform distribution of stationary electrons to cancel out the charge of the ions, that is

$$\left.\begin{aligned} n &= n_0 \qquad \text{given by} \quad -en_0 + \rho_0 = 0 \\ \vec{u} &= 0 \\ \vec{E} &= 0 \\ \vec{B} &= 0. \end{aligned}\right\} \text{This satisfies all fluid + Maxwell eqns.}$$

To linearize, write

$$\left.\begin{aligned} n &= n_0 + n_1 \\ \vec{u} &= \vec{u}_1 \\ \vec{E} &= \vec{E}_1 \\ \vec{B} &= \vec{B}_1 \end{aligned}\right\}$$

then substitute, keep only terms through 1st order.

$$\frac{\partial}{\partial t}(\cancel{n_0} + n_1) + \nabla\cdot(n_0\vec{u}_1) = 0$$

$$m\left(\frac{\partial \vec{u}_1}{\partial t} + 0\right) = -e\left(\vec{E}_1 + 0\right)$$

$$\nabla\cdot\vec{E}_1 = \frac{1}{\epsilon_0}\left(-e(\cancel{n_0}+n_1) + \cancel{\rho_0}\right)$$

or clean it up,

$$\left.\begin{aligned} \frac{\partial n_1}{\partial t} + n_0\,\nabla\cdot\vec{u}_1 &= 0 \\ m\frac{\partial\vec{u}_1}{\partial t} &= -e\vec{E}_1 \\ \nabla\cdot\vec{E}_1 &= -\frac{e}{\epsilon_0}n_1 \end{aligned}\right\}$$ Linearized eqns.

<u>Step 3</u> Do F.T. in time, $\partial/\partial t \to -i\omega$:

$$\left.\begin{aligned} -i\omega n_1 + n_0\,\nabla\cdot\vec{u}_1 &= 0 \\ -i\omega m\,\vec{u}_1 &= -e\vec{E}_1 \\ \nabla\cdot\vec{E}_1 &= -\frac{e}{\epsilon_0}n_1 \end{aligned}\right\}.$$

<u>Step 4</u> Get $\vec{P}$ in terms of $\vec{E}$: (or $\vec{P}_1, \vec{E}_1$)

Let $\vec{\xi}$ = electron displacement

$$\vec{u} = \dot{\vec{\xi}} \quad \text{or} \quad \vec{u} = -i\omega\vec{\xi}.$$

$$\vec{P} = -en_0\vec{\xi} = (-en_0)\left(\frac{i}{\omega}\right)\left(-\frac{ie}{m\omega}\right)\vec{E}_1 = -\frac{n_0e^2}{m\omega^2}\vec{E}_1$$

$$\vec{\xi} = \frac{i}{\omega}\vec{u}_1 \qquad\qquad = \epsilon_0\chi_e(\omega)\vec{E}_1$$

$$\vec{u}_1 = -\frac{ie}{m\omega}\vec{E}_1$$

Mon. 12/2/02.

Hence $\chi_e(\omega) = - \frac{n_0 e^2}{m \epsilon_0} \frac{1}{\omega^2} = - \frac{\omega_p^2}{\omega^2}$,

$$\boxed{\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right).}$$ for plasmas in this cold electron model.

This means for $\omega < \omega_p$, $\epsilon < 0$. Since $k^2 = \omega^2 \mu_0 \epsilon(\omega)$,

it means k = imaginary, and waves in this ω range do not propagate into a plasma. This explains the reflection of radio waves from the ionosphere.

Reflection of light (optical freq) from conductors is similar ($\epsilon < 0$) but more complicated in detail since resonances are important.

Now we consider the Kramers-Kronig relations, which concern the analyticity properties of $\epsilon(\omega)$. These are very general, and depend only on causality and not on any specifics of the properties of the medium. Thus they apply to any causal, linear relationship between fields. Very useful in particle and solid state physics.

Start with $\vec{D} = \epsilon\vec{E} = \epsilon_0(1+\chi_e)\vec{E} = \epsilon_0\vec{E} + \vec{P}$,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

Both ϵ, χ_e are operators, $\epsilon = \epsilon_0(1+\chi_e)$, only trivial difference between them so look at χ_e.

Assuming static, local, isotropic model, in t-domain we have

$$\vec{P}(\vec{x},t) = \epsilon_0 \int_{-\infty}^{t} dt' \; \chi_e(t-t') \, \vec{E}(\vec{x},t').$$

Upper limit is t by causality, no effect from $t'>t$, or can say $\chi_e(t-t')=0$ for $t'>t$. Now set ~~t-t'~~ $\tau = t-t'$, τ = time before present, and

$$\vec{P}(\vec{x},t) = \epsilon_0 \int_0^{\infty} d\tau \; \chi_e(\tau) \, \vec{E}(\vec{x},t-\tau).$$

Here $\chi_e(\tau) = 0$ for $\tau<0$, by causality. Then $\chi_e(\omega)$ is F.T. of $\chi_e(\tau)$, in the sense

$$\chi_e(\omega) = \int_0^{\infty} d\tau \; e^{i\omega\tau} \chi_e(\tau) \qquad (\text{w/o } \sqrt{2\pi}).$$

Then

$$\vec{P}(\vec{x},\omega) = \chi_e(\omega)\, \vec{E}(\vec{x},\omega) \qquad (\text{convolution thm})$$

where

$$\vec{E}(\vec{x},\omega) = \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}} e^{i\omega t} \vec{E}(\vec{x},t) \qquad (\text{with } \sqrt{2\pi})$$

sim. for $\vec{P}$. Jackson writes $G(\tau)$ instead of $\chi_e(\tau)$, better because abuse of notation to use same symbol χ_e for 2 functions, $\chi_e(\tau)$ and $\chi_e(\omega)$.