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Summary Polarization of Light.

$\vec{E}(\vec{x},t) = \mathrm{Re}\left[\vec{a}\, e^{i(kz-\omega t)}\right]$ most general plane wave with $\vec{k} = k\hat{z}$.

where $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ = complex 2-vector.

equal or proportional to, won't matter for this subject.

intensity $= |\vec{a}|^2 = |a_x|^2 + |a_y|^2 = \langle \chi | \chi \rangle$ where $|\chi\rangle = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$.

$\vec{a}$ = 2-spinor. This is classical E+M, but quantum formalism of spin 1/2 particle is useful.

If we restrict ourselves to measurements of intensity, then the overall phase of the field cannot be measured, That is,

$$|\chi\rangle \to e^{i\varphi}|\chi\rangle \qquad \varphi = \text{a phase}$$

makes no difference in intensity measurements. (Similar to Q.M.)

But relative phase between 2 components of $\vec{a}$ or $|\chi\rangle$ can be measured.

Note that $|\chi\rangle \to e^{i\varphi}|\chi\rangle$ is equivalent to shifting the origin of time,

$$kz - \omega t + \varphi = kz - \omega t' \quad \text{where} \quad t' = t - \varphi/\omega.$$

Since we average over rapid time scale $2\pi/\omega$ in an intensity measurement, ~~we cannot see~~ the phase shift is not observable.

But we can transform or filter the light before making a measurement of intensity. Using a linear polarizer, we can project out the light polarized in the $\hat{x}$- or $\hat{y}$- or any direction $\hat{n}$ in the x-y plane.

$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{operator that projects out } x \text{ polarization.}$$

If $|\chi\rangle \to P_x|\chi\rangle$, i.e. $\begin{pmatrix} a_x \\ a_y \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} a_x \\ 0 \end{pmatrix}$, Mon. 11/25/02

then the intensity after the measurement is

$$\langle \chi | P_x^+ P_x | \chi \rangle = \langle \chi | P_x | \chi \rangle = |a_x|^2$$

since $P_x = P_x^+ = P_x^2$. Similarly,

$$P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$ projects out lin. polarized light in y direc.

real unit vector

$$P_{\hat{n}} = \begin{pmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{pmatrix}$$ projects lin. pol. light in $\hat{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$ direc.

Note, $\langle \chi | P_{\hat{n}} | \chi \rangle = | n_x a_x + n_y a_y |^2 = |\hat{n} \cdot \vec{a}|^2$.

Another operation you can do before measuring light is to apply a <u>1/4-wave plate</u>. This is an anisotropic medium of the right thickness to cause the light in one direction (say, x) to be phase shifted by $\pi/2$ (1/4 wave) relative to the other (say, y) direction.

One of these is called the "fast" and the other the "slow" direction. If the fast direction is x and the slow one y, then the operator representing the 1/4 wave plate is

$$Q_x = \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}.$$

We can just as well write

$$Q_x = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$ (this is more symmetrical)

since the overall phase doesn't matter. Q is a unitary operator.

We can orient the (fast axis) 1/4 wave plate in other directions, and obtain other operators, Q_y, $Q_{\hat{n}}$.

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The 2-vector $\binom{a_x}{a_y}$ or $|\chi\rangle$ has 4 real parameters, to describe the state of the light, but if we restrict ourselves to intensity measurements then ~~they~~ there are on 3 parameters since we can't measure the overall phase. These 3 real parameters are usually taken to be the Stoke's parameters s_1, s_2, s_3, defined by

$$s_2 = \langle \chi | \sigma_1 | \chi \rangle$$
$$s_3 = \langle \chi | \sigma_2 | \chi \rangle$$
$$s_1 = \langle \chi | \sigma_3 | \chi \rangle$$

where σ_i, $i = 1,2,3$ are the Pauli matrices. The indices on s_i are not the same as those on σ_i because of old conventions.

One can show that the Stokes' parameters can be measured by intensity measurements combined with linear ~~project~~ filters $P_{\hat{n}}$ and/or 1/4 wave plates $Q_{\hat{n}}$. One can also show that knowledge of (s_1, s_2, s_3) implies knowledge of $\vec{a}$ (the complex 2-vector) to within an overall phase.

One also defines a 4-th Stokes' parameter,

$$s_0 = \langle \chi | \chi \rangle ,$$

but it is not indep. of the others: You can show that

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 .$$

Thus, the state of polarization is described by a point on a sphere of radius s_0. This is called the <u>Poincaré Sphere</u>.

All of this applies to a light wave of definite frequency ω. In practice you always have a spread $\Delta\omega$ in freq. Imagine light from some source (the sun, a star, a laser, etc). If $\Delta\omega \ll \omega$ you have <u>quasi-monochromatic</u> light. For sunlight or starlight it will be necessary to use a frequency selecter (prism, diffraction grating etc). to select some $\Delta\omega$ interval.

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Since quasimonochromatic light is not exactly at one frequency, the parameters $\vec{a}$ of the light evolve in time on the scale $\frac{2\pi}{\Delta\omega} \gg \frac{2\pi}{\omega}$.

However, $\frac{2\pi}{\Delta\omega}$ may still be extremely short time. E.g. with sunlight if $\frac{\Delta\omega}{\omega} = 10^{-3}$, you have $\frac{2\pi}{\Delta\omega} \sim 10^{-12}$ or 10^{-13} sec. Thus intensity $\langle\chi|\chi\rangle$ fluctuates on time scale $\frac{2\pi}{\Delta\omega}$.

Thus, if you make a measurement of intensity in which you average over a time scale longer than $\frac{2\pi}{\Delta\omega}$, you get $\overline{\langle\chi|\chi\rangle}$, where the overbar — means <u>time average</u> on scale longer than $\frac{2\pi}{\Delta\omega}$.

Or, if you insert ~~a~~ linear x-polarizer before the intensity measurement, you get $\overline{\langle\chi|P_x|\chi\rangle}$.

Averaging on a time scale ~~t~~ $\gg$ $\frac{2\pi}{\Delta\omega}$ is the same as averaging over an <u>ensemble</u> of measurements, each made at one instant of time but separated by times $\gg \frac{2\pi}{\Delta\omega}$.

So we define an operator,

$$\rho = \overline{|\chi\rangle\langle\chi|} = \overline{\begin{pmatrix} |a_x|^2 & a_x^* a_y^* \\ a_x^* a_y^* & |a_y|^2 \end{pmatrix}}$$

the <u>density operator</u>.

This is also a <u>correlation matrix</u>

and the average value of, say P_x, is

$$\overline{\langle\chi|P_x|\chi\rangle} = \mathrm{tr}\left(P_x\, \overline{|\chi\rangle\langle\chi|}\right) = \mathrm{tr}\,(P_x\, \rho),$$

and similarly for other operators P_y etc.

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If you measure the Stokes' parameters for a statistical ensemble described by ρ, you find in general,

$$\mathrm{tr}(\rho\sigma_i) = \overline{\langle x|\sigma_i|x\rangle} = \begin{Bmatrix} S_2 \\ S_3 \\ S_1 \end{Bmatrix} \qquad i=1,2,3.$$

and

$$S_0 = \mathrm{tr}\,\rho = \overline{\langle x|x\rangle},$$

you find

$$S_1^2+S_2^2+S_3^2 \le S_0^2.$$

Thus, the state of polarization is described by a vector $\vec{S}$ that lies on or in the interior of the Poincaré sphere (with radius S_0).

Purely polarized light gives $\vec{S}$ on the surface of the sphere, while unpolarized light has $\vec{S}=0$. Intermediate cases are possible too.

The vector $\vec{\alpha}$ or $|x\rangle$ above has been represented in the xy or linear polarization basis, $\hat{\varepsilon}_x = \hat{x}$, $\hat{\varepsilon}_y = \hat{y}$. But we can transform to other bases, such as the $\pm$ helicity bases,

$$\hat{\varepsilon}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}, \qquad \hat{\varepsilon}_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

(or LCP) (or RCP)

These are complex unit vectors, they are orthonormal in the sense

$$\hat{\varepsilon}_i^{*}\cdot\hat{\varepsilon}_j = \delta_{ij} \qquad i,j=1,2.$$

This is the beginning of the statistical mechanics of the EM field.