EM wave propagation in media. 1

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Model: Static, local, region-wise uniform, isotropic medium.

(E.g., glass, air, but not too high ω).

$$\Rightarrow \quad \epsilon(\omega) = \text{F.T. of } \epsilon(t,t') = \epsilon(t-t').$$

and $\vec{D}(\vec{x},\omega) = \epsilon(\omega)\,\vec{E}(\vec{x},\omega)$. also $\vec{B}(\vec{x},\omega) = \mu(\omega)\,\vec{E}(\vec{x},\omega)$

Write out source free Maxwell's eqns, all fields are fns. of $(\vec{x},\omega)$ (they have been F.T.'d in time, e.g.

$$\vec{E}(\vec{x},\omega) = \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t}\, \vec{E}(\vec{x},t).$$

Note, $\frac{\partial}{\partial t} \rightarrow -i\omega$. $\rho = \vec{J} = 0$.

$$\nabla\cdot\vec{D} = \nabla\cdot(\epsilon\vec{E}) = 0$$

$$\nabla\cdot\vec{B} = \nabla\cdot(\mu\vec{H}) = 0.$$

$$\nabla\times\vec{E} = i\omega\vec{B} = i\omega\mu\vec{H}$$

$$\nabla\times\vec{H} = -i\omega\vec{D} = -i\omega\epsilon\vec{E}$$

Now ϵ depends on ω inside a given region. Can say it depends on $\vec{x}$ at a boundary betw. regions, where it jumps. So inside a given region, ϵ indep. of $\vec{x}$ and $[\nabla, \epsilon] = 0$. Sim $[\nabla, \mu] = 0$

So, $\nabla\cdot(\epsilon\vec{E}) = \epsilon\,\nabla\cdot\vec{E} = 0 \Rightarrow \nabla\cdot\vec{E} = 0$.

and, $\nabla\times(\nabla\times\vec{E}) = \nabla(\cancel{\nabla\cdot\vec{E}}) - \nabla^2\vec{E} = i\omega\mu\,\nabla\times\vec{H} = \omega^2\epsilon\mu\vec{E}$.

So, $\boxed{(\nabla^2 + \epsilon\mu\,\omega^2)\vec{E}(\vec{x},\omega) = 0}$ Helmholz eqn.

Note, $\epsilon = \epsilon(\omega)$, $\mu = \mu(\omega)$ in general, and in general these are complex

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<u>Define</u> $$n(\omega)^2 = \frac{\epsilon(\omega)\,\mu(\omega)}{\epsilon_0 \mu_0} = c^2\,\epsilon(\omega)\mu(\omega).$$

$n(\omega) =$ <u>index</u> of <u>refraction</u>. Note it may be complex too.

So can write Helmholtz equ in form,

$$\left[\nabla^2 + \frac{n(\omega)^2\,\omega^2}{c^2}\right]\vec{E}(\vec{x},\omega) = 0.$$

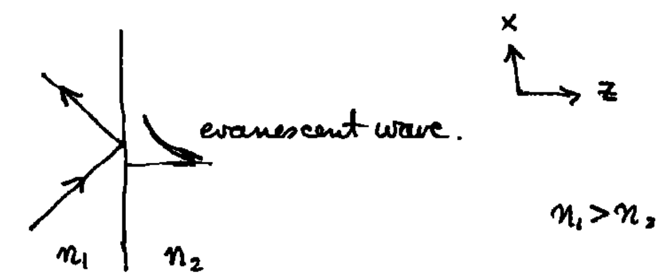
Helmholtz equ. possesses plane wave solutions,

$$\vec{E}(\vec{x},\omega) = \vec{E}_0\, e^{i\vec{k}\cdot\vec{x}}$$

$\vec{E}_0$ is a (generally complex) amplitude. Plugging in we find

$$k^2 = \vec{k}\cdot\vec{k} = \frac{n^2\omega^2}{c^2}.$$

Note, if $n(\omega)$ complex then $\vec{k}$ <u>must</u> be complex; but $\vec{k}$ <u>may</u> be complex even if $n(\omega)$ is real. For example, in total internal reflection,



In less dense medium, if $\vec{k} = k_x\hat{x} + i\kappa_z\hat{z}$, $\kappa_z =$ real, >0

<u>also</u>, $\nabla\cdot\vec{E} = 0 \Rightarrow$
$\vec{k}\cdot\vec{E}_0 = 0$
$\vec{k}\perp\vec{E}_0$.

$$k^2 = k_x^2 - \kappa_z^2 = \frac{n^2\omega^2}{c^2}.$$

$$e^{i(k_x\hat{x} + i\kappa_z\hat{z})\cdot\vec{x}} = e^{ik_x x}\,e^{-\kappa_z z}$$

Complex $\vec{k}$ normally not allowed in ∞ medium (it blows up exponentially in some direction).

General solution of Helmholtz eqn at fixed ω is a linear combination of solutions of form $\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x})}$

(an integral actually) taken over all $\vec{k}$ that satisfy $k^2 = n^2\omega^2/c^2$ including complex $\vec{k}$'s, if applicable (allowed by bdry cond's). General solution of any kind (not restricting ω) is then a linear combination over ω. But must require $\vec{E}(\vec{x},\omega) = \vec{E}(\vec{x},-\omega)^*$ so that $\vec{E}(\vec{x},t) = $ real.

alternatively, if you are only interest in particular solutions of fixed ω, multiply soln of Helmholz eqn by $e^{-i\omega t}$ and take Re part.

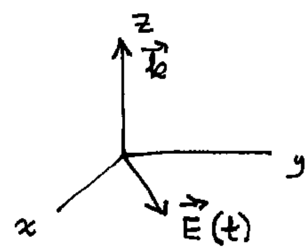
For a plane wave soln, $\vec{E}(\vec{x},\omega) = \vec{E}_0\, e^{i\vec{k}\cdot\vec{x}}$, get $\vec{B}$ from Faraday's law,

$$\vec{B} = -\frac{i}{\omega}\nabla\times\vec{E} = \frac{1}{\omega}\vec{k}\times\vec{E}_0\, e^{i\vec{k}\cdot\vec{x}}$$
$$= \vec{B}_0\, e^{i\vec{k}\cdot\vec{x}}$$

where $\vec{B}_0 = \frac{1}{\omega}\vec{k}\times\vec{E}_0$.

- Now polarization of light.

In the following we consider a plane, electromagnetic wave propagating in the z-direction, with definite frequency ω. It can be either in a medium or in vaccuum, but if in a medium we will assume $n =$ real.



Then $\vec{k} = k\hat{z}$, $\vec{E}$ eval. at $z=0$ lies in the x-y plane.

First suppose $\vec{E}$ is in x-direction. Then most general form for $\vec{E}$ is

$$\vec{E}(\vec{x},t) = \hat{x}\, A_x \cos(kz - \omega t + \alpha_x)$$

where $A_x \geq 0$ is a real amplitude and α_x is a phase. At optical frequencies it is difficult to measure α_x, since it's the same as knowledge of origin of time ($t=0$ of the clock) to within an optical period $2\pi/\omega \sim 10^{-15}$ sec. Involves phase info about the wave.

Write this as
$$\vec{E}(\vec{x},t) = \mathrm{Re}\left(\hat{x}\, a_x\, e^{i(kz-\omega t)}\right)$$

where
$$a_x = A_x e^{i\alpha_x} = \text{complex amplitude.}$$

Notice, complex wave has same information in it as real wave.

Similarly, the most gen. form of wave with $\vec{E}$ in y-direc. is
$$\mathrm{Re}\left(\hat{y}\, a_y\, e^{i(kz-\omega t)}\right).$$

So most general EM wave of freq. ω propagating in z-direc is

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left[\vec{a}\, e^{i(kz-\omega t)}\right] = \text{lin. comb. of 2 waves above.}$$

where $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ is the complex amplitude vector.

There are 4 parameters in specifying such a wave.

<u>Examples</u>: $\vec{a} = E_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $E_0 = \text{real}, > 0$ <u>linear polarizn in x-direc.</u>

$\vec{a} = E_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, lin. polariz. in y-direc.

Called linear because $\vec{E}(z=0,t)$ traces out a straight line.

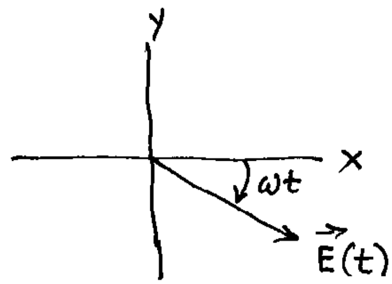
<u>Example:</u> $\vec{a} = E_0 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$, $E_0 > 0$ real.

$-i = e^{-i\pi/2}$

Then
$$\left.\begin{aligned} E_x &= \frac{E_0}{\sqrt{2}} \cos(kz - \omega t) \\ E_y(\vec{x},t) &= \frac{E_0}{\sqrt{2}} \cos(kz - \omega t - \pi/2) \end{aligned}\right\} \text{ eval at } z=0.$$

$$E_x = \frac{E_0}{\sqrt{2}} \cos\omega t$$
$$E_y = \frac{E_0}{\sqrt{2}} \cos(\omega t + \pi/2) = -\frac{E_0}{\sqrt{2}} \sin\omega t.$$

In x-y plane:
look at $\vec{E}$ as fn. of t.



$\vec{E}$ rotates in circle <u>clockwise</u>. This is <u>R</u>ight <u>C</u>ircularly <u>P</u>olarized light. (RCP)

Sim., if $\vec{a} = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}$ we get L.C.P. light.

In particle physics talk about <u>helicity</u>, $\vec{S}\cdot\hat{p}$ projection of spin onto direction of motion. Then: $\left.\begin{aligned} \text{RCP} &= \text{helicity } -1 \\ \text{LCP} &= \text{helicity } +1 \end{aligned}\right\}$.

Types of measurement you can do on light wave:

1. Phase. Requires knowing $\vec{E}(t)$ on time scale of $2\pi/\omega$. Difficult to do at high ω.

2. Intensity Easy to do at any ω. Intensity is proportional to $|\vec{E}(\vec{x},t)|^2$. at high ω, you normally measure only average intensity (averaged over oscillations, that is, time-averaged on scale $\frac{2\pi}{\omega}$).

In the following we will look at intensity measurements only. Then overall phase of wave cannot be measured, $\vec{a} \to e^{i\varphi}\vec{a}$, can't measure φ.

pol. in x same as pol. in -x

Total intensity of wave,

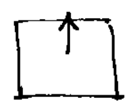
$$\vec{E}(\vec{x},t) = \mathrm{Re}\left[\vec{a}\, e^{i(kz-\omega t)}\right]$$

measured at fixed $\vec{x}$ (say $\vec{x}=0$), is proportional to $|\vec{a}|^2 = |a_x|^2 + |a_y|^2$.

Define a 2-component "spinor" $|\chi\rangle = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$. Intensity $= \langle\chi|\chi\rangle$

~~Intens.~~ $|\chi\rangle \to e^{i\varphi}|\chi\rangle$ shift in origin of time

Now can measure intensity of x or y components by using ~~filter~~ linear polarizers.



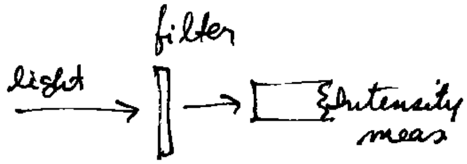
piece of material, passes light polarized in one direc. only.

You can rotate it in any direction.

In x-direc., input $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$, output $\vec{a} = \begin{pmatrix} a_x \\ 0 \end{pmatrix}$.

Equiv. to projection operator,
$$P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Sim, in y-direc.,
$$P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$



Intensity if you filter in x or y is

$\langle\chi|P_x|\chi\rangle$

or $\langle\chi|P_y|\chi\rangle$.