summary

$$D_i(\vec{x},t) = \sum_j \int d^3\vec{x}'\,dt' \quad \epsilon_{ij}(\vec{x},t;\vec{x}',t')\; E_j(\vec{x}',t') \qquad \text{(general)}$$

$$D_i(\vec{x},t+\tau) = \sum_j \int d^3\vec{x}'\,dt' \quad \epsilon_{ij}(\vec{x},t;\vec{x}',t')\; E_j(\vec{x}',t'+\tau) \qquad \begin{matrix}\forall\tau\\ \text{(static medium)}\end{matrix}$$

$$\Rightarrow \quad D_i(\vec{x},t) = \sum_j \int d^3\vec{x}'\,dt' \quad \epsilon_{ij}(\vec{x},t-\tau;\vec{x}',t'-\tau)\; E_j(\vec{x}',t')$$

$$\Rightarrow \quad \epsilon_{ij}(\vec{x},t-\tau;\vec{x}',t'-\tau) = \epsilon_{ij}(\vec{x},t;\vec{x}',t') \qquad \forall\tau.$$

$$\Rightarrow \quad \epsilon_{ij}(\vec{x},t;\vec{x}',t') = \left(\text{fn. of } t-t' \text{ only}\right) = \epsilon_{ij}(\vec{x},\vec{x}';t-t')$$

Result of translational invariance in time.

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Similarly, if we have a uniform medium, then $\epsilon_{ij}(\vec{x},t;\vec{x}',t')$ depends only on $\vec{x}-\vec{x}'$. So, for a static, uniform medium,

$$\epsilon_{ij}(\vec{x},t;\vec{x}',t') = \epsilon_{ij}(\vec{x}-\vec{x}',t-t').$$

A uniform medium is a strong assumption; it's only a model, but it means all of space is filled with glass (for example), we can talk about nondissipatives waves in glass but not reflection or refraction which require an interface between 2 media.

Another assumption we might make is that of an isotropic medium, which says that the properties are the same in all directions. Then

$$\epsilon_{ij}(\vec{x}t;\vec{x}'t') = \delta_{ij}\ \epsilon(\vec{x}t;\vec{x}'t'),$$

so ϵ becomes a scalar.

Consider a uniform, static medium, so that

$$D_i(\vec{x},t) = \sum_j \int d^3\vec{x}'dt' \quad \epsilon_{ij}(\vec{x}-\vec{x}',t-t')\ E_j(\vec{x}',t').$$

Introduce Fourier transforms of $\vec{E},\vec{D}$,

$$\vec{E}(\vec{x},t) = \int \frac{d^3\vec{k}\,d\omega}{(2\pi)^2}\ e^{i(\vec{k}\cdot\vec{x}-\omega t)}\ \vec{E}(\vec{k},\omega)$$

and inverse

$$\vec{E}(\vec{k},\omega) = \int \frac{d^3\vec{x}\,dt}{(2\pi)^2}\ e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\ \vec{E}(\vec{x},t),$$

and similarly for $\vec{D}(\vec{x},t)$, $\vec{D}(\vec{k},\omega)$. Notice that $\vec{E}(\vec{x},t)$ and $\vec{E}(\vec{k},\omega)$ are really different functions (more abuse of notation).

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Then you find,

$$\boxed{\tilde{D}_i(\vec{k},\omega) = \sum_j \epsilon_{ij}(\vec{k},\omega)\, \tilde{E}_j(\vec{k},\omega)}$$ static, uniform.

where

$$\epsilon_{ij}(\vec{k},\omega) = \int d^3\vec{x}\, dt \quad \epsilon_{ij}(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}.$$

This only works for a static, uniform medium, but in that case, the action of ϵ in $(\vec{k},\omega)$-space is purely multiplicative.

This is the <u>convolution</u> (or Faltung) theorem.

If the medium is static but not uniform, then we only do the FT in time, and we get ↳ maybe only piece-wise uniform

$$D_i(\vec{x},\omega) = \sum_j \int d^3\vec{x}' \quad \epsilon_{ij}(\vec{x},\vec{x}',\omega) \quad E_j(\vec{x}',\omega)$$

where $\vec{E}(\vec{x},\omega)$ is the time F.T. of $\vec{E}(\vec{x},t)$, etc.

Another assumption we can make is <u>locality</u>. This means that $\vec{D}(\vec{x})$ depends only on the value of $\vec{E}$ at the <u>same</u> <u>point</u> $\vec{x}$. Then ϵ must be proportional to $\delta^3(\vec{x}-\vec{x}')$,

$$\epsilon_{ij}(\vec{x},t;\vec{x}',t') = \delta^3(\vec{x}-\vec{x}')\; \epsilon_{ij}(\vec{x},t,t'),$$

so

$$D_i(\vec{x},t) = \int dt' \sum_j \epsilon_{ij}(\vec{x},t,t')\; E_j(\vec{x},t').$$

Some people might include in the definition of "local" the possibility that $\vec{D}$ at $\vec{x}$ could depend on the value of $\vec{E}$ at $\vec{x}$ and on the spatial derivatives of $\vec{E}$ at $\vec{x}$. Then ~~Eij would~~ $\epsilon_{ij}(\vec{x}t;\vec{x}'t')$ would also involve derivatives of δ-fns, $\partial_i \delta^3(\vec{x}-\vec{x}')$, etc.

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If a medium is both local and static then we can F.T. in time and get

$$D_i(\vec{x},\omega) = \sum_j \epsilon_{ij}(\vec{x},\omega)\; E_j(\vec{x},\omega). \qquad (*)$$

If the medium is local, ~~stat~~ and uniform, ~~then~~ and static, then

$$\epsilon_{ij}(\vec{x}t;\vec{x}'t') = \delta^3(\vec{x}-\vec{x}')\,\underbrace{\epsilon_{ij}(\vec{x};t,t')} \qquad (\text{local})$$

$$= \epsilon_{ij}(t,t') \qquad (\text{uniform})$$

$$= \epsilon_{ij}(t-t') \qquad (\text{static}).$$

So in this case we can F.T. in time to get

$$D_i(\vec{x},\omega) = \sum_j \epsilon_{ij}(\omega)\, E_j(\vec{x},\omega).$$

But ~~local~~ uniform assumption is often too strong. Let's assume that medium is uniform in regions (piecewise uniform) like glass-vacuum. Then if static and local, we have $(*)$ above, where $\epsilon_{ij}(\vec{x},\omega)$ is indep. of $\vec{x}$ in a given region. Then we have

$$\vec{D}(\vec{x},\omega) = \sum_j \epsilon_{ij}^{(R)}(\omega)\, \vec{E}(\vec{x},\omega),$$

where R labels which region we're in. Finally, if medium is also isotropic, we have

$$\vec{D}(\vec{x},\omega) = \epsilon^{(R)}(\omega)\, \vec{E}(\vec{x},\omega).$$

This is the model Jackson is using in Sec. 7.1 of the book (static, local, piece-wise or region-wise uniform, isotropic).

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Now look at microscopic model for $\epsilon(\omega)$. Will see how assumptions of locality come in. This is Jackson Sec. 7.5.

Bound electron on atom, posn $\vec{x} = \vec{x}_0 + \vec{\xi}$, $\vec{x}_0$ = atomic posn. Assume bound by a harmonic oscillator of freq. ω_0. Eqn. of motion,

$$m\left(\ddot{\vec{\xi}}(t) + \underbrace{\gamma\,\dot{\vec{\xi}}(t)}_{\text{damping term}} + \omega_0^2\,\vec{\xi}(t)\right) = -e\,\underbrace{\vec{E}(\vec{x}_0+\vec{\xi},t)}_{\text{approx. } \vec{E}(\vec{x}_0,t)}$$

This is where locality comes in

Do F.T. in time, apply $\int \frac{dt}{\sqrt{2\pi}} e^{i\omega t}$, get

$$m\left(-\omega^2 - i\gamma\omega + \omega_0^2\right)\vec{\xi}(\omega) = -e\,\vec{E}(\vec{x}_0,\omega),$$

(really F.T. of)

Thus $\vec{p}(\omega)$ = dipole moment $= -e\,\vec{\xi}(\omega)$

$$= \frac{e^2}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\,\vec{E}(\vec{x}_0,\omega).$$

Let:

N = # atoms/vol.

Z = # electrons/atom.

f_j = # electrons with freq. ω_j, so $\sum_j f_j = Z$.

Then: $\vec{P}(\vec{x}_0,\omega) = \dfrac{\text{dipole mom}}{\text{vol.}} = \left[\dfrac{Ne^2}{m}\sum_j \dfrac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}\right]\vec{E}(\vec{x}_0,\omega).$

Can drop 0 on $\vec{x}_0$ now.

But $\vec{P} = \epsilon_0\chi_e\vec{E}$, $\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon\vec{E}$.

So,

$$\boxed{\epsilon(\omega) = \epsilon_0\left[1 + \frac{Ne^2}{\epsilon_0 m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}\right]}$$

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Notice, $\epsilon^{(\omega)}$ is complex, but $\epsilon(-\omega) = \epsilon(\omega)^*$ (since it is the F.T. of a real qty.)

This is a crude model. Describes bound electrons only. Ignores Quantum mechanics, although quantum results are similar, where ω_j are the energies for excitation into excited states. Damping γ_j is normally

over ħ

$\ll \omega_j$, so imaginary part of $\epsilon(\omega)$ is important only near resonances.

If ω is well away from any resonant freq ω_j, $|\omega-\omega_j| \gg \gamma_j$, then $\epsilon(\omega)$ is approximately real.

Note, this model ignores difference betw. macroscopic avg $\vec{E}$ and microscopic $\vec{e}$ at posn of electron. You can improve the model with the kind of reasoning that went into Clausius Mossotti eqn.

Will analyze $\epsilon(\omega)$ more later, for now just wanted to have a simple example of how you might compute $\epsilon(\omega)$.

Now look at (source-free) wave propagation in a homogeneous medium, when $\epsilon = \epsilon(\omega)$, $\mu = \mu(\omega)$. This is straightfwd. This could be one region of a piece-wise uniform medium.
First F.T. Max. eqns. in t, so all fields become fns of $(\vec{x},\omega)$ and $\partial/\partial t \to -i\omega$. Note, if $\epsilon = \epsilon(\omega)$ (indep. of $\vec{x}$) then ϵ commutes with ∇.

$$\left.\begin{aligned}
\nabla\cdot\vec{D} &= \nabla\cdot(\epsilon\vec{E}) = \epsilon\,\nabla\cdot\vec{E} = 0.\\
\nabla\cdot\vec{B} &= 0.\\
\nabla\times\vec{E} &= +i\omega\vec{B} = i\omega\mu\vec{H}\\
\nabla\times\vec{H} &= -i\omega\vec{D} = -i\omega\epsilon\vec{E}.
\end{aligned}\right\}.$$

$$\Rightarrow\quad \nabla\times(\nabla\times\vec{E}) = \nabla(\cancelto{0}{\nabla\cdot\vec{E}}) - \nabla^2\vec{E} = i\omega\mu\,\nabla\times\vec{H} = \omega^2\epsilon\mu\vec{E}.$$

So, $\vec{E}(\vec{x},\omega)$ satisfies, $\boxed{(\nabla^2 + \epsilon\mu\omega^2)\vec{E}(\vec{x},\omega) = 0}$ Helmholtz eqn.