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Summary.

$$\left.\begin{aligned}\vec{E} &= e\,\frac{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right]}{cR\,(1-\hat{n}\cdot\vec{\beta})^3}\\ \vec{B} &= \hat{n}\times\vec{E}\end{aligned}\right\}\ \text{radii fields.}$$

Factors of γ:

1. For fixed F = force, $\dfrac{P_\perp}{P_\parallel} \sim \gamma^2$ (power radiated due to $\perp$ accel. vs. $\parallel$ accel.)

2.  $\Delta\theta \sim \frac{1}{\gamma}$ radiation lobe.

3. Circular motion, $\omega \sim \gamma^3\omega_0$  etc.

Power spectrum: (Think of synchrotron radiation from astrophysical source.)

$$\frac{d^2W}{dt\,d\Omega} = R^2\,\vec{S}\cdot\hat{n} = \frac{c}{4\pi}R^2\left|\vec{E}(t)\right|^2 = \left|\vec{A}(t)\right|^2$$

where $\vec{A}(t) = \sqrt{\dfrac{c}{4\pi}}\;R\,\vec{E}(t)$.

$$\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} dt\left|\vec{A}(t)\right|^2 = \int_{-\infty}^{+\infty} d\omega\,\left|\vec{A}(\omega)\right|^2 = 2\int_0^\infty d\omega\,\left|\vec{A}(\omega)\right|^2 = \int_0^\infty d\omega\,\frac{d^2W}{d\omega\,d\Omega}.$$

where $\vec{A}(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} dt\; e^{i\omega t}\,\vec{A}(t)$.

$$\frac{d^2W}{d\omega\,d\Omega} = 2\left|\vec{A}(\omega)\right|^2.$$ "Differential power spectrum"

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$$\vec{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\; e^{i\omega t} \sqrt{\frac{c}{4\pi}}\; \underbrace{R\,\vec{E}(t)}$$

$$\longrightarrow \frac{R\,e}{cR} \left\{ \frac{\hat{n} \times [(\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}]}{(1-\hat{n}\cdot\vec{\beta})^3} \right\}_{ret} . \quad \text{R's cancel.}$$

Fourier transform taken w.r.t. field time, but $\vec{E}$ involves retarded time. Switch to retarded time (t') as var. of integration.

Use $\quad \dfrac{dt}{dt'} = 1 - \hat{n}\cdot\vec{\beta}$, so kills one power in denom.

$$\vec{A}(\omega) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{c}{4\pi}}\; \frac{e}{c} \int_{-\infty}^{+\infty} dt'\; e^{i\omega t} \; \frac{\hat{n} \times [(\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}]}{(1-\hat{n}\cdot\vec{\beta})^2}$$

But $\quad t = t' + R/c, \quad R = |\vec{x} - \vec{r}(t')|$.

Here $\vec{r}(t') =$ orbit of particle (above called $\vec{y}$). Suppose this is measured wrt an origin centered on small arc of the orbit from which radiation is received. Then $|\vec{r}| \lll R$, and

$$R = |\vec{x} - \vec{r}(t)| = R_0 - \hat{n}\cdot\vec{r}, \qquad R_0 = |\vec{x}|.$$

$$e^{i\omega t} = e^{i\omega(t' + \frac{R_0}{c} - \frac{\hat{n}\cdot\vec{r}}{c})} = e^{i\omega\frac{R_0}{c}}\; e^{i\omega(t' - \frac{\hat{n}\cdot\vec{r}(t')}{c})} .$$

$$e^{\frac{i\omega R_0}{c}} = \text{const. phase}$$

So...

$$= \sqrt{\frac{e^2}{8\pi^2 c}} \int_{-\infty}^{+\infty} dt\; e^{i\omega(t - \frac{\hat{n}\cdot\vec{r}}{c})} \; \underbrace{\frac{\hat{n} \times [(\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}]}{(1-\hat{n}\cdot\vec{\beta})^2}}$$

dropping prime on t. Now use

$$= \frac{d}{dt} \left(\frac{\hat{n} \times (\hat{n} \times \vec{\beta})}{1 - \hat{n}\cdot\vec{\beta}} \right)$$

and integrate by parts. Notice,

$$\frac{d}{dt}\left(t - \frac{\hat{n}\cdot\vec{r}}{c}\right) = 1 - \hat{n}\cdot\vec{\beta}.$$

Gives

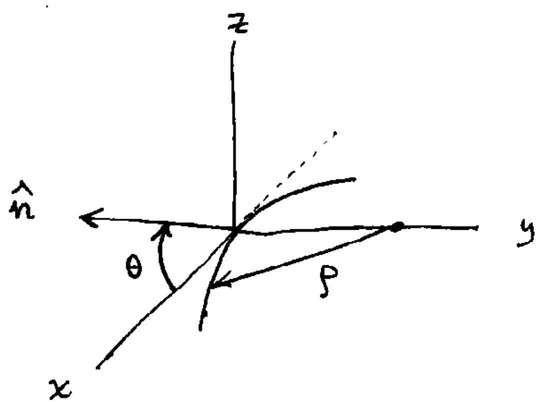
$$\vec{A}(\omega) = \sqrt{\frac{e^2}{8\pi^2 c}}\ \underbrace{\left(-i\, e^{i\omega R_0/c}\right)}_{\text{const a phase, will drop out when we square.}}\ \omega \int_{-\infty}^{+\infty} dt\ e^{i\omega\left(t - \frac{\hat{n}\cdot\vec{r}}{c}\right)}\ \hat{n}\times(\hat{n}\times\vec{\beta}).$$

+ bdry terms

↑ argue away

Integral is nominally from $-\infty$ to $+\infty$, but really only small orbit segment is important.

So just need to do integral. Introduce coordinates. Put orbital plane in XY plane, let observer lie in XZ-plane (far away).



ρ = radius of curvature

Let $\omega_0 = \frac{v}{\rho}$ = freq. of circular motion if particle really had a circular orbit.

So,

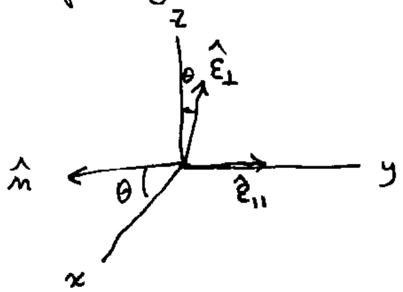
$$\hat{n} = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}, \quad \vec{r}(t) = \rho \begin{pmatrix} \sin\omega_0 t \\ (1-\cos\omega_0 t) \\ 0 \end{pmatrix}$$

$$\vec{\beta}(t) = \frac{\rho\omega_0}{c}\begin{pmatrix} \cos\omega_0 t \\ \sin\omega_0 t \\ 0 \end{pmatrix}$$

$$\frac{\rho\omega_0}{c} \rightarrow \frac{v}{c} = \beta$$

We do not assume orbit is a circle, but we are only interested in a small part of the orbit which can be approximated by a circle.

Introduce polarization vectors, $\hat{\varepsilon}_{\parallel}$ in plane of orbit
$\hat{\varepsilon}_{\perp}$ perp to this.



$$\hat{\varepsilon}_{\parallel} = \hat{y}$$

$$\hat{\varepsilon}_{\perp} = \begin{pmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{pmatrix}.$$

Then: $\quad \hat{n} \times (\hat{n} \times \vec{\beta}) = \beta \sin\theta \cos\omega_0 t \; \hat{\varepsilon}_{\perp} - \beta \sin\omega_0 t \; \hat{\varepsilon}_{\parallel}$.

Write $\vec{A}(\omega) = A_{\perp}(\omega)\,\hat{\varepsilon}_{\perp} + A_{\parallel}(\omega)\,\hat{\varepsilon}_{\parallel}$.

$$\left.\begin{aligned} A_{\perp}(\omega) &= \binom{\text{const}}{\text{phase}} \sqrt{\frac{e^2}{8\pi^2 c}}\ \beta\omega \sin\theta \int_{-\infty}^{+\infty} dt\ e^{i\varphi(t)} \cos\omega_0 t \\ A_{\parallel}(\omega) &= \binom{\text{const}}{\text{phase}} \sqrt{\frac{e^2}{8\pi c^2}}\ \beta\omega \int_{-\infty}^{+\infty} dt\ e^{i\varphi(t)} \sin\omega_0 t \end{aligned}\right\}$$

where $\varphi(t) = \omega\left(t - \frac{\hat{n}\cdot\vec{r}}{c}\right)$.

Now approximate integrals, based on $\beta \approx 1$ i.e. $\gamma \gg 1$, $\theta \ll 1$, arc of orbit $\omega_0 t$ integrated over is small ($\omega_0 t \ll 1$).

In fact, small parameter is $\frac{1}{\gamma} \sim \theta \sim \omega_0 t$.

> Must approximate exponent to higher order than rest of integrand, bec. ω may be large, gives rapidly oscillating phase.

First approx $\varphi(t)$: $\quad \frac{\hat{n}\cdot\vec{r}}{c} = \frac{\rho}{c} \cos\theta \sin\omega_0 t = \frac{\beta}{\omega_0}\cos\theta\sin\omega_0 t$.

$$\frac{\rho}{v}\frac{v}{c} = \frac{\beta}{\omega_0}$$

So, $\omega t - \frac{\omega\,\hat{n}\cdot\vec{r}}{c} = \omega t - \frac{\omega}{\omega_0}\beta\cos\theta\sin\omega_0 t$

$$= \omega t - \frac{\omega}{\omega_0}\left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right)\left(\omega_0 t - \frac{\omega_0^3 t^3}{6}\right)$$

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$$\varphi(t) = \frac{\omega t}{2}\left(\frac{1}{\gamma^2} + \theta^2\right) + \frac{\omega\omega_0^2}{6} t^3 .$$

Standard integrals,

$$Ai(\eta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dz \; e^{i(z\eta + z^3/3)}$$

<u>Airy fns.</u>

$$Ai'(\eta) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} dz \; z \; e^{i(z\eta + z^3/3)} .$$

In above let $\left(\frac{\omega\omega_0^2}{2}\right)^{1/3} t = z$,

Then $\varphi(t) = z\eta + \frac{z^3}{3}$, where $\boxed{\eta = \left(\frac{\omega}{2\omega_0}\right)^{2/3}\left(\frac{1}{\gamma^2} + \theta^2\right)}$.

Gives

$$\left\{ \begin{matrix} A_\perp(\omega) \\ A_\parallel(\omega) \end{matrix} \right\} = \begin{pmatrix} \text{const} \\ \text{phase} \end{pmatrix} \sqrt{\frac{e^2}{2c}} \times \left\{ \begin{matrix} \theta \left(\frac{2\omega^2}{\omega_0^2}\right)^{1/3} Ai(\eta) \\ \left(\frac{4\omega}{\omega_0}\right)^{1/3} Ai'(\eta) \end{matrix} \right\}$$

Relation to Jackson: $\xi = \frac{2}{3}\eta^{3/2} = \frac{\omega}{3\omega_0}\left(\frac{1}{\gamma^2} + \theta^2\right)^{3/2}$

$$Ai(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}} \; K_{1/3}\left(\tfrac{2}{3} z^{3/2}\right)$$

$$Ai'(z) = \frac{1}{\pi}\frac{z}{\sqrt{3}} \; K_{2/3}\left(\tfrac{2}{3} z^{3/2}\right)$$

So,

$$\frac{d^2W}{d\omega\, d\Omega} = 2\left(|A_\perp|^2 + |A_\parallel|^2\right)$$

$$\left(\frac{d^2W}{d\omega\, d\Omega}\right)_{\perp,\parallel} = \frac{e^2}{c} \times \left\{ \begin{matrix} \theta^2 \left(\frac{2\omega^2}{\omega_0^2}\right)^{2/3} Ai^2(\eta) & (\perp) \\ \left(\frac{4\omega}{\omega_0}\right)^{2/3} Ai'(\eta)^2 & (\parallel). \end{matrix} \right.$$