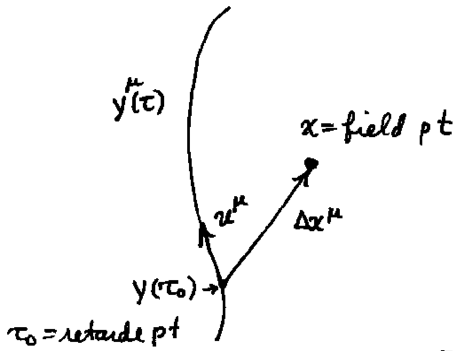
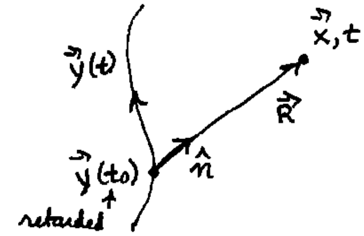
Wed 11/6/02

Summary.

(c=1 in 4D formulas
c restored in 3+1 formulas) (final)



4D picture



3D picture

Defns.

$$\Delta x^\mu = x^\mu - y^\mu(\tau_0) \qquad \Delta x^\mu = \begin{pmatrix} R \\ \vec{R} \end{pmatrix}$$

$$\Delta x \cdot \Delta x = 0 \quad (\text{defn of } \tau_0)$$

$$u^\mu = \frac{dy^\mu}{d\tau} \qquad \vec{\beta} = \vec{v}/c \qquad \vec{R} = \vec{x} - \vec{y}(\tau_0)$$

$$b^\mu = \frac{du^\mu}{d\tau} \qquad \dot{\vec{\beta}} = \vec{a}/c \qquad R = |\vec{R}| \qquad \hat{n} = \frac{\vec{R}}{R}.$$

$$\frac{\partial \tau_0}{\partial x_\mu} = \frac{\Delta x_\mu}{(\Delta x \cdot u)}$$

$$\boxed{A^\mu(x) = \left.\frac{e u^\mu}{(\Delta x \cdot u)}\right|_{\tau_0}}$$

Liénard-Wiechert

or

$$\boxed{\begin{aligned} \Phi(\vec{x}, t) &= \frac{e}{R(1-\vec{\beta}\cdot\hat{n})} \\ \vec{A}(\vec{x}, t) &= \frac{e\vec{\beta}}{R(1-\vec{\beta}\cdot\hat{n})} \end{aligned}}$$

$$\Delta x \cdot u = c\gamma R(1 - \vec{\beta}\cdot\hat{n}) \quad (c \neq 1)$$

$$\text{or } \gamma R(1 - \vec{v}\cdot\hat{n}) \quad (c=1)$$

$$\left(\frac{\partial t_{field}}{\partial t_{ret}}\right)_{\vec{x}} = 1 - \vec{\beta}\cdot\hat{n}$$

$$\boxed{F^{\mu\nu}(x) = e\left\{\frac{\Delta x^\mu b^\nu - \Delta x^\nu b^\mu}{(\Delta x \cdot u)^2} - \frac{\Delta x^\mu u^\nu - \Delta x^\nu u^\mu}{(\Delta x \cdot u)^3}(\Delta x \cdot b - 1)\right\}_{\tau = \tau_0}}$$

$$\boxed{\begin{aligned} \vec{E} &= e\left\{\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{c R (1 - \hat{n}\cdot\vec{\beta})^3} + \frac{\hat{n} - \vec{\beta}}{\gamma^2 R^2 (1 - \hat{n}\cdot\vec{\beta})^3}\right\} \\ \vec{B} &= \hat{n} \times \vec{E} \end{aligned}}$$

That is, work in a frame in which particle is at rest at the retarded posn.

(2) Wed 11/6/02

Would like to compute power radiated by accel. charge. First take easy case, where $\vec{v}=0$. Then (only looking at acceleration fields)

this means, at retarded time.

$$\vec{E} = \frac{e}{c}\,\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R} = \frac{e}{c}\,\frac{\hat{n}(\hat{n}\cdot\dot{\vec{\beta}})-\dot{\vec{\beta}}}{R} = -\hat{n}\times\vec{B}.$$

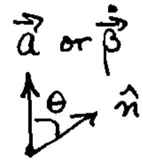
$$\vec{B} = \hat{n}\times\vec{E} = \frac{e}{c}\,\frac{-(\hat{n}\times\dot{\vec{\beta}})}{R}$$

$$\vec{S} = \text{Poynting vector} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \cancel{\frac{e^2}{4\pi c}}$$

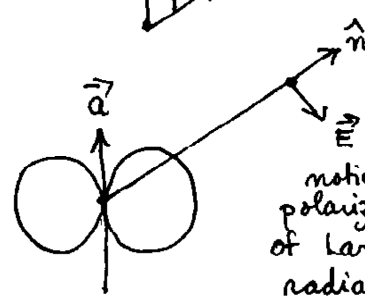
$$= \frac{c}{4\pi}\,\vec{B}\times(\hat{n}\times\vec{B}) = \frac{c}{4\pi}\,\hat{n}\,B^2$$

$$= \frac{e^2}{4\pi c R^2}\,\hat{n}\,(\hat{n}\times\dot{\vec{\beta}})^2$$

$$= \frac{e^2}{4\pi c R^2}\,\hat{n}\,\left(\dot{\beta}^2-(\hat{n}\cdot\dot{\vec{\beta}})^2\right)$$



$$= \frac{e^2}{4\pi c}\,\dot{\beta}^2\frac{\sin^2\theta}{R^2}\,\hat{n}$$



notice polarization of Larmor radiation.

Gives $\sin^2\theta$ radiation pattern. Radiation preferentially goes $\perp$ to acceleration.

really over surf. of sphere centered on retarded pt.

Integrate this over all solid angles, note R^2 cancels,

$$P = \int R^2 d\Omega \cdot \frac{e^2}{4\pi c}\,\dot{\beta}^2\frac{\sin^2\theta}{R^2} =$$

$\int d\Omega \sin^2\theta = \frac{8\pi}{3}$, so...
$$\boxed{P = \frac{2}{3}\,\frac{e^2 a^2}{c^3}}$$

Larmor formula, valid for $\vec{v}=0$, approx. valid for $v \ll c$.

Many applications.

means, at retar

③

Wed 11/6/02

Would like to find power radiated when $\vec{v} \neq 0$. Hard way to do this is to use general formula for $\vec{E}, \vec{B}$ and integrate $\vec{S}$ over surface of sphere. (In fact, there's a subtlety, even if you do it this way... later).

field time vs retarded time

Easy way is to use Lorentz covariance. Set $c = 1$ for now. Then write power radiated per unit time in retarded rest frame as

$$\frac{\Delta E}{\Delta t} = \frac{2}{3} e^2 a^2 .$$

Now compute momentum radiated per unit time, still in retarded rest frame (RRF). Do this by integrating the stress tensor over the surface of a sphere of radius R centered on retarded point.

You find: $$\frac{\Delta \vec{p}}{\Delta t} = 0. \qquad \text{(Reasonable from radiation pattern).}$$

Now, in RRF $\quad \Delta t = \Delta t_{field} = \Delta t_{ret} \quad$ since $\left(\frac{\partial t_{field}}{\partial t_{ret}}\right)_{\vec{x}} = 1 - \vec{\beta}\cdot\hat{n} = 1$ in RRF

and $\Delta t_{ret} = \Delta \tau_0$ (τ_0 = retarded proper time), so,

$$\frac{1}{\Delta \tau_0} \begin{pmatrix} \Delta E \\ \Delta \vec{p} \end{pmatrix} = \frac{2}{3} e^2 a^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{in RRF.}$$

↑

This is not Δp^μ of particle, it is Δp^μ of radiation emitted in $\Delta \tau_0$.

But $u^\mu = \frac{dy^\mu}{d\tau_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in RRF. Also,

$$b^\mu b_\mu = b\cdot b = -\gamma^6 (\vec{v}\cdot\vec{a})^2 - \gamma^4 a^2 = -a^2 \quad \text{in RRF,}$$

Wed 11/6/02

So, $$\boxed{\frac{dp^\mu}{d\tau_0} = -\frac{2}{3} e^2 (b\cdot b)\, u^\mu}$$

As derived, this eqn is only valid in RRF. But we expect $\Delta p^\mu =$ 4-momentum radiated in $\Delta\tau_0$ to transform as a 4-vector, and u^μ certainly transforms as a 4-vector, and coef. $(b\cdot b)$ is a Lorentz scalar, so whole eqn. must be valid in any frame. So switch to any frame now.

Now take $\mu = 0$ component,

$$\frac{dE}{d\tau_0} = \frac{2}{3} e^2 \left[\gamma^6 (\vec{v}\cdot\vec{a})^2 + \gamma^4 a^2 \right] \frac{dt_{ret}}{d\tau_0}$$

or $$\frac{dE}{dt_{ret}} = \frac{2}{3}\frac{e^2}{c} \left[\gamma^6 (\vec{\beta}\cdot\dot{\vec{\beta}})^2 + \gamma^4 \dot{\beta}^2 \right]$$ (restoring factors of c)

or $$\boxed{\frac{dE}{dt_{ret}} = \frac{2}{3}\frac{e^2}{c}\gamma^6 \left[\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2 \right]}$$ (Jackson 14.26, but he doesn't make it clear that it's power w.r.t. retarded time.)

Generalization of Larmor's formula to arb. velocities. Notice, ~~energy~~ power indicated is energy per unit retarded time. You might have thought we should look at ~~power~~ energy per unit field time, that is a different quantity and harder to compute. So if you want to compute integral of $\vec{S}$ over sphere you can do it, but should integrate

$$\int_{\text{sphere}} \underbrace{\left(\frac{dt_{field}}{dt_{ret}}\right)}_{\text{extra factor of } 1-\vec{\beta}\cdot\hat{n}} \vec{S}\cdot d\vec{a}$$ if you want to get answer above.

Now specialize to case $\vec{v} \parallel \vec{a}$ (or $\vec{\beta} \parallel \dot{\vec{\beta}}$).

$$\frac{dE}{dt} \left(\text{means } \frac{dE}{dt_{ret}}\right) = \frac{2}{3}\frac{e^2}{c}\gamma^6 \dot{\beta}^2 = \frac{2}{3}\frac{e^2}{c^3}\gamma^6 a^2 \text{ (parallel case).}$$

Express in terms of force, $F = m\gamma^3 a$ (for $\parallel$ force).

$$\frac{dE}{dt} = \frac{2}{3}\frac{e^2}{m^2c^3}F^2 = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2 \qquad \text{since } F = \frac{dE}{dx} = \frac{1}{v}\frac{dE}{dt}$$

But have to be careful, $\frac{dE}{dt}$ refers to radiated energy, $\frac{dE}{dx}$ refers to particle energy. So, interesting ratio is

$$\left(\frac{\frac{dE_{rad}}{dt}}{\frac{dE_{part.}}{dt}}\right) = \frac{2}{3}\frac{e^2}{m^2c^3v}\frac{dE}{dx} \qquad E = E_{part.}$$

You try to accelerate a particle, but some of the energy goes into radiation. What is fractional loss? Introduce:

$$r_0 = \frac{e^2}{mc^2} = \text{"classical radius of particle"}$$
$$= 2.8 \times 10^{-13} \text{ cm for electron.}$$

mc^2 = rest energy of particle.

$$\epsilon = \frac{E}{mc^2}, \quad \xi = \frac{x}{r_0} \qquad \text{dimensionless variables.}$$

$$= \frac{2}{3}\frac{1}{\beta}\frac{d\epsilon}{d\xi}.$$

So suppose $\beta \sim 1$ and you want

$$\frac{dE_{rad}/dt}{dE/dt} \sim 1.$$

$$\Rightarrow \frac{d\epsilon}{d\xi} \sim 1, \quad \frac{dE}{dx} = \frac{mc^2}{e^2/mc^2} = \frac{500\text{KeV}}{2\times10^{-13}\text{cm}}.$$