Mon 11/4/02

Now Jackson's $\boxed{\text{covariant Green's functions}}$. These are only slight variations on $G_\pm$ (advanced, retarded) Green's fns. Will mainly use retarded G-fn.

Remind, $\underbrace{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{-\Box = -\partial_\mu \partial^\mu} G_\pm(\vec{x},t;\vec{x}',t') = -4\pi\, \delta(\vec{x}-\vec{x}')\underbrace{\delta(t-t')}_{c\,\delta(x^0-x'^0)}$

So $\quad \Box G_\pm(x,x') = 4\pi c\, \delta^4(x-x') \qquad x=(ct,\vec{x}), \quad x'=(ct',\vec{x}')$

Now, D_r (retarded), D_a (advanced) defined by $\boxed{\Box D = \delta^4(x-x')}$

So, $\quad D_r = \frac{1}{4\pi c} G_+$

Now, express D_r (or G_+) in covariant form.

Recall, $\quad G_+(\vec{x},t;\vec{x}',t') = \dfrac{\delta(t'-t+R/c)}{R}, \quad R = |\vec{x}-\vec{x}'|$

Let $(x-x')^2 = (x-x')\cdot(x-x') = c^2(t-t')^2 - R^2$, and consider

$\delta((x-x')^2)$. regard as fn of t'. Has roots at $t' = t \pm R/c$.

Need $\quad \left|\frac{d}{dt'}(x-x')^2\right| = \left|2c^2(t'-t)\right| = \frac{2R}{c}c^2 = 2Rc$

So, $\quad \delta((x-x')^2) = \dfrac{\delta(t'-t-R/c)}{2Rc} + \dfrac{\delta(t'-t+R/c)}{2Rc}$

corresponds to $\uparrow$ advanced $\qquad \uparrow$ retarded.

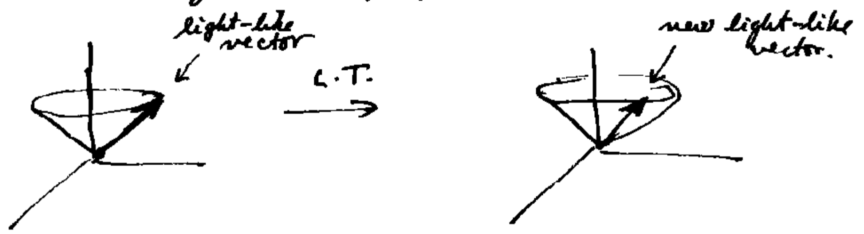
Want to kill advanced, so multiply by $\Theta(t-t')$. $\qquad \Theta(x) \equiv \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$

So $\quad \Theta(t-t')\,\delta((x-x')^2) = \dfrac{\delta(t'-t+R/c)}{2Rc} = \dfrac{1}{2c} G_+$

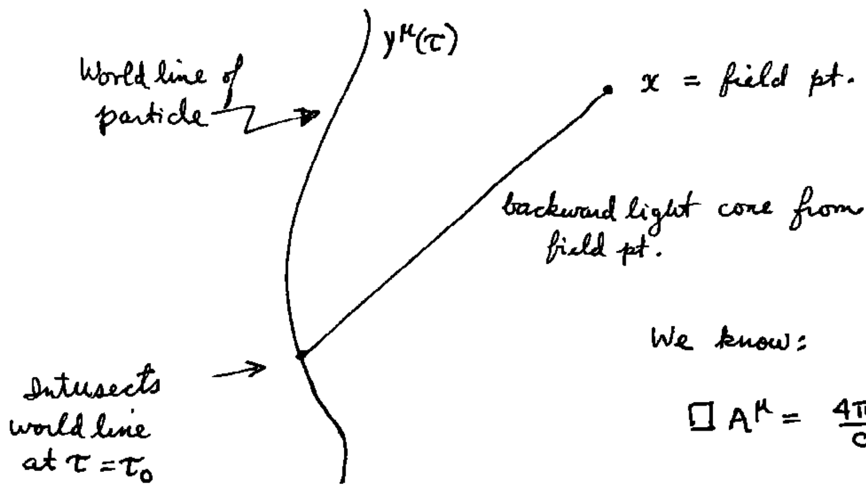
Mon. 11/4/02

So, $$\boxed{D_r(x-x') = \frac{1}{2\pi}\,\Theta(t-t')\,\delta\big((x-x')^2\big)}$$

Covariant version of propagator. $\Theta(t-t')$ doesn't look covariant (it isn't by itself) but is covariant when multiplied by $\delta((x-x')^2)$, which forces interval $x-x'$ to be light-like. The time ordering of 2 events separated by time-like or light-like interval is invariant under a proper Lorentz transformation, because fwd light cone is mapped into itself under proper L.T.



Now use this green's fn to get potentials of a particle in arbi. state of motion.



We know:

$$\Box A^\mu = \frac{4\pi}{c} J^\mu.$$

Hence $$A^\mu(x) = \frac{4\pi}{c}\int d^4x' \, D_r(x-x')\, J^\mu(x'),$$

But $$J^\mu(x') = ce\int d\tau \frac{dy^\mu}{d\tau}\,\delta^4(x'-y(\tau))$$

So $$= 4\pi e\int d^4x' \,\frac{1}{2\pi}\,\Theta(t-t')\,\delta\big((x-x')^2\big)\int d\tau\, \frac{dy^\mu}{d\tau}\,\delta^4(x'-y(\tau))$$

$$= 2e\int d\tau\, \Theta(t-t')\,\delta\big((x-y(\tau))^2\big)\,\frac{dy^\mu}{d\tau}$$

Mon Nov. 4 2002

Need root in τ of $(x-y(\tau))^2=0$ with $t>t'$. Geometrically, this is intersection of backward light cone from field pt. ~~to~~ with world line of particle. Say this occurs at $\tau=\tau_0$.

$$\text{need } \left|\frac{d}{d\tau}(x-y(\tau))^2\right| = \left|2(x-y(\tau))\cdot\frac{dy}{d\tau}\right| \text{ eval @ } \tau=\tau_0.$$

Note, $2(x-y(\tau_0))\cdot\frac{dy}{d\tau}(\tau_0)>0$ (eval. in rest frame at τ_0).

So, $$\boxed{A^\mu(x) = \left.\frac{e\,u^\mu}{u\cdot(x-y)}\right|_{\tau=\tau_0}} \quad \text{where } u=\frac{dy}{d\tau}.$$

Lienard-Wiechert potentials.

Put L-W potentials in 3+1 notation

Let $\Delta x^\mu = x^\mu - y^\mu(\tau_0)$ 4-vector from ~~f~~ retarded source to field pt.

or $\Delta x^\mu = (R, \vec{R})$,

where $\vec{R} = \vec{x}-\vec{y}(\tau_0)$, 3-vector from retarded source to field pt.

$R = |\vec{R}|$.

Note $\Delta x^2 = \Delta x^\mu \Delta x_\mu = 0$ (defn of τ_0).

Then $u^\mu = \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix}$ so $u\cdot(x-y) = u\cdot\Delta x = c\gamma R - \gamma\vec{v}\cdot\vec{R}$

$$= c\gamma R(1-\hat{n}\cdot\vec{\beta})$$

where $\hat{n} = \frac{\vec{R}}{R}$ = unit vector from retarded source to field pt

$\vec{\beta} = \vec{v}/c$ Jackson's dimensionless velocity.

Summary: $$\boxed{u\cdot(x-y) = u\cdot\Delta x = c\gamma R(1-\hat{n}\cdot\vec{\beta})}$$

Mon 11/4/02

So, since $A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}$, we get

$$\boxed{\begin{aligned} \Phi(\vec{x},t) &= \frac{e}{R(1-\hat{n}\cdot\vec{\beta})} \\ \vec{A}(\vec{x},t) &= \frac{e\vec{\beta}}{R(1-\hat{n}\cdot\vec{\beta})} \end{aligned}}$$

where RHS eval. at retarded pt, e.g. $R = |\vec{x}-\vec{y}(\tau_0)|$, etc.

Lienard-Wiechert potentials in 3+1 notation.

These give correct results in low velocity, small retardation limit.

Now compute EM fields from L-W potentials. This is just algebra, but have to be careful because A^μ contains an <u>explicit</u> dependence on x (through $x-y(\tau_0)$ in denom) and <u>implicit</u> one thru τ_0.

First implicit dependence.

$$0 = \Delta x\cdot\Delta x = (x-y(\tau_0))\cdot(x-y(\tau_0)).$$

Apply $\partial_\mu = \frac{\partial}{\partial x^\mu}$:
$$0 = 2(x_\mu - y_\mu(\tau_0)) + 2(x_\nu - y_\nu(\tau_0))\Big(-\underbrace{\frac{dy_\nu}{d\tau_0}}_{= u^\nu}\Big)\frac{\partial\tau_0}{\partial x^\mu}$$

or, solving,
$$\frac{\partial\tau_0}{\partial x^\mu} = \frac{\Delta x_\mu}{\Delta x\cdot u}.$$

$\cancel{\partial_\mu(\Delta x\cdot u) = u_\mu + (\Delta x\cdot b - u\cdot u)}$

Thus if
$$A^\mu(x) = \frac{e u^\mu}{\Delta x\cdot u},$$

$\cancel{\partial_\mu(\Delta x\cdot u) =}$

$$\partial_\mu A_\nu = e\left\{\underbrace{-\frac{u_\nu}{(\Delta x\cdot u)^2}u_\mu}_{\text{explicit}} + \underbrace{\overbrace{\left[\frac{b_\nu}{\Delta x\cdot u} - \frac{u_\nu}{(\Delta x\cdot u)^2}(\Delta x\cdot b - u^2)\right]}^{\frac{d}{d\tau_0}\left(\frac{u_\nu}{\Delta x\cdot u}\right)}\overbrace{\frac{\Delta x_\mu}{(\Delta x\cdot u)}}^{\frac{\partial\tau_0}{\partial x^\mu}}}_{\text{implicit.}}\right\}$$

Here $b^\mu = \left.\frac{du^\mu}{d\tau}\right|_{\tau_0}$, and $u^2 = u\cdot u = c^2$.

Mon. 11/4/02

So,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\boxed{F^{\mu\nu} = e\left\{ \frac{\Delta x^\mu b^\nu - \Delta x^\nu b^\mu}{(\Delta x\cdot u)^2} - \frac{\Delta x^\mu u^\nu - \Delta x^\nu u^\mu}{(\Delta x\cdot u)^3}\,(\Delta x\cdot b - c^2)\right\}}$$

Now put into 3+1 notation.

$$\frac{d\gamma}{dt} = \gamma^3 \frac{\vec{v}\cdot\vec{a}}{c^2} = \gamma^3 \vec{\beta}\cdot\dot{\vec{\beta}}$$

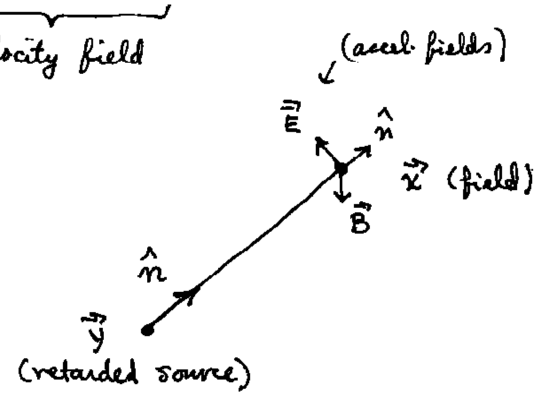
$$b^\mu = \frac{du^\mu}{d\tau} = \gamma \frac{d}{dt}\begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix}$$

$$= \gamma \begin{pmatrix} \gamma^3 \vec{\beta}\cdot\dot{\vec{\beta}}\, c \\ \gamma^3 \vec{\beta}\cdot\dot{\vec{\beta}}\,\vec{v} + \gamma\vec{a} \end{pmatrix} = \begin{pmatrix} \gamma^4 \vec{\beta}\cdot\dot{\vec{\beta}}\, c \\ \gamma^4 \vec{\beta}\cdot\dot{\vec{\beta}}\,\vec{v} + \gamma^2\vec{a} \end{pmatrix}.$$

Do the algebra. $F^{i0} = E_i$, find $F^{ij} = -\varepsilon_{ijk} B_k$

$$\vec{E} = e\left\{ \underbrace{\frac{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right]}{cR\,(1-\hat{n}\cdot\vec{\beta})^3}}_{\text{acceleration field.}} + \underbrace{\frac{(\hat{n}-\vec{\beta})}{\gamma^2 R^2\,(1-\hat{n}\cdot\vec{\beta})^3}}_{\text{velocity field}} \right\}$$

$$\vec{B} = \hat{n}\times\vec{E}.$$



- velocity field $\sim \frac{1}{R^2}$.

accel. field $\sim \frac{1}{R}$, dominates at large R, $\vec{E}, \vec{B}, \hat{n}$ make right-handed systems. $\vec{E}, \vec{B}$ equal in magnitude.