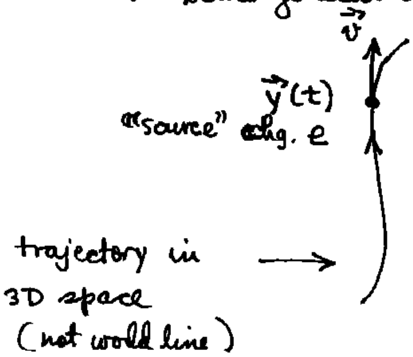
Today we do $\boxed{\text{Darwin Lagrangian}}$ which describes EM interaction between particles to lowest orders in v/c. It is a nonrelativistic approximation $+ (v/c)^2$ corrections. Thus it is specific to one Lorentz frame and for the most part we will not try to be covariant.

First some general observations.



• $\vec{x}$ = field pt.

If source is stationary, then

$$\Phi(\vec{x}) = \frac{e}{|\vec{x}-\vec{y}|}.$$

If source is moving, then has source current $\vec{J}(\vec{x},t) = e\,\vec{v}\,\delta^3(\vec{x}-\vec{y})$.

So must be a magnetic field. where $\vec{v} = \dfrac{d\vec{y}}{dt}$.

What is $\vec{B}$ at field pt?

A guess is to use Biot-Savart law,

$$\vec{B}(\vec{x}) = \frac{1}{c}\int d^3\vec{x}'\; \vec{J}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \frac{e}{c}\,\vec{v} \times \frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}.$$

But Biot-Savart law is only valid for steady currents, so this is a guess. But in fact it's correct to lowest order in v/c (not exact).

Note: $\vec{B} \sim v/c$, $\vec{F}_{mag} \sim (v/c)^2$.

Now put a 2nd particle at field pt $\vec{x}$. We must change notation to avoid confusion.

source	field
$\vec{y} \to \vec{x}_1$	$\vec{x} \to \vec{x}_2$
$\vec{v} \to \dot{\vec{x}}_1 \equiv \vec{v}_1$	$\vec{v}_2 = \dot{\vec{x}}_2$
$e \to e_1$	e_2 = chg at field pt.

Then

Mag.
Force on e_2 due to e_1 $= \frac{e_2}{c}\vec{v}_2 \times \vec{B}(\vec{x}_2)$

$$\vec{F}_{21}^{mag} = \frac{e_1 e_2}{c^2} \quad \vec{v}_2 \times \left[\vec{v}_1 \times \frac{(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^3}\right]$$

clearly of order $\left(\frac{v}{c}\right)^2$.

$$= \frac{e_1 e_2}{c^2}\left[\vec{v}_1\, \vec{v}_2\cdot(\vec{x}_2 - \vec{x}_1) - (\vec{x}_2 - \vec{x}_1)\,\vec{v}_1\cdot\vec{v}_2\right]\frac{1}{|\vec{x}_2 - \vec{x}_1|^3}.$$

By swapping 1,2 we get

Mag force on e_1 due to e_2 =

$$\vec{F}_{12}^{mag} = \frac{e_1 e_2}{c^2}\left[\vec{v}_2\;\; \vec{v}_1\cdot(\vec{x}_1 - \vec{x}_2) - (\vec{x}_1 - \vec{x}_2)\,\vec{v}_1\cdot\vec{v}_2\right]\frac{1}{|\vec{x}_2 - \vec{x}_1|^3}.$$

add them, get

$$\vec{F}_{12}^{mag} + \vec{F}_{21}^{mag} = \frac{e_1 e_2}{c^2} \; \frac{(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^3} \times (\vec{v}_1 \times \vec{v}_2) \qquad \neq 0.$$

Newton's 2nd law not satisfied. This only applies to magnetic forces, but if you add the electric forces these also do not satisfy Newton's 2nd law.

Of course the dominant force is due to the electrostatic field,

$$\vec{E}(\vec{x}_2) = -\nabla_2\left(\frac{e_1}{|\vec{x}_2 - \vec{x}_1|}\right), \qquad \text{and } 1 \Leftrightarrow 2,$$

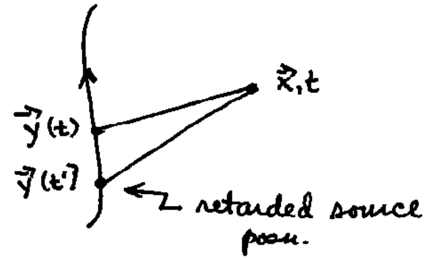
and these do satisfy Newton's 2nd law.

How to do systematic expansion for low velocities.

Go back to $\vec{x}, \vec{y}$ notation

$$\underset{\text{field}}{\vec{x}},\; \underset{\text{source}}{\vec{y}}$$

Exact soln's involve retarded times.



In Lorenz gauge,

$$\Phi(\vec{x},t) = \int d^3\vec{x}' \, \frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|}$$

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3\vec{x}' \, \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}$$

where t' = retarded time $= t - \underbrace{\frac{|\vec{x}-\vec{x}'|}{c}}$.

small correction term if retardation small.

Let $R \equiv |\vec{x}-\vec{x}'|$.

$$\rho(\vec{x}',t') = \rho\left(\vec{x}', t-\frac{R}{c}\right) = \rho(\vec{x}',t) - \frac{R}{c}\frac{\partial}{\partial t}\rho(\vec{x}',t) + \frac{R^2}{2c^2}\frac{\partial^2}{\partial t^2}\rho(\vec{x}',t) + \cdots$$

Sim $\vec{J}(\vec{x}',t') = \vec{J}(\vec{x}',t) - \frac{R}{c}\frac{\partial}{\partial t}\vec{J}(\vec{x}',t) + \cdots$ etc.

So, $$\Phi(\vec{x},t) = \int d^3\vec{x}' \, \frac{\rho(\vec{x}',t)}{R} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\underbrace{\int d^3\vec{x}' \, \rho(\vec{x}',t)}_{\text{total chg. } Q = \text{const.}}} + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int d^3\vec{x}' \rho(\vec{x}',t) R + \cdots$$

Now, $\rho(\vec{x}',t) = e\,\delta^3(\vec{x}'-\vec{y}(t))$, so can do all integrals,

$$\Phi(\vec{x},t) = \frac{e}{R} + \frac{e}{2c^2}\frac{\partial^2}{\partial t^2} R + \cdots$$ Thru $1/c^2$

Sim. $$\vec{A}(\vec{x},t) = \frac{e}{c}\frac{\vec{v}}{R} + \cdots$$ Leading term only.

where now $R = |\vec{x}-\vec{y}(t)|$.

Now, correction term in Φ involves 2nd time deriv's. of R which will involve accel $= \frac{d^2\vec{y}}{dt^2}$ of source particle. Later will want to ~~do~~ use in a

Lagrangian, where we don't want accelerations (only velocities).
So to get rid of these, do a gauge transformation,

$$\left.\begin{aligned} \Phi' &= \Phi - \frac{1}{c}\frac{\partial g}{\partial t} \\ \vec{A}' &= \vec{A} + \nabla g \end{aligned}\right\} \quad \text{choose } g = \frac{e}{2c}\frac{\partial}{\partial t}R \quad \text{to kill 2nd term in } \Phi.$$

Then $\Phi(\vec{x},t) = \dfrac{e}{R}$

$$\vec{A}(\vec{x},t) = \frac{e}{c}\frac{\vec{v}}{R} + \frac{e}{2c}\nabla\left(\frac{\partial}{\partial t}R\right) \qquad \text{(do the algebra)}$$

$$= \frac{e}{2c}\left[\frac{\vec{v}}{R} + \frac{\vec{R}(\vec{v}\cdot\vec{R})}{R^3}\right] \qquad \text{where } \vec{R} = \vec{x} - \vec{y}(t).$$

Notice that $\Phi = \dfrac{e}{|\vec{x}-\vec{y}(t)|}$ now, which is unretarded Coulomb potential.

Thus, gauge transf. has taken us from Lorenz to Coulomb gauge.

We could have solved for the potentials directly in Coulomb gauge (and expanded in pwrs of $1/c$), but this requires us to compute the transverse part of the current. That is what Jackson does. The present approach is easier and cleaner IMHO.

Notice also, if we compute the magnetic field (for this it is easier to work in Lorenz gauge where $\vec{A}$ is simpler) we get

$$\vec{B} = \nabla\times\vec{A} = \frac{e}{c}\nabla\times\left(\frac{\vec{v}}{R}\right) = \frac{e}{c}\vec{v}\times\frac{\vec{R}}{R^3},$$

which is naive result from Biot-Savart law.

Now change notation again to

source	field
$\vec{y} \to \vec{x}_1$	$\vec{x} \to \vec{x}_2$
$e \to e_1$	$e_2 =$ 2nd chg.
$\vec{v} \to \vec{v}_1 = \dot{\vec{x}}_1$	$\vec{v}_2 = \dot{\vec{x}}_2$

also, put

$\vec{R} = \vec{x} - \vec{y} \to \vec{x}_2 - \vec{x}_1 \equiv \vec{r}_{21} = -\vec{r}_{12}$

$R = |\vec{x} - \vec{y}| \to r_{21} = r_{12}$

Then interaction Lagrangian of 2nd particle in field of 1st is...

$$-e_2\left(\Phi(\vec{x}_2) - \frac{\vec{v}_2}{c}\cdot\vec{A}(\vec{x}_2)\right)$$

$$= -\frac{e_1 e_2}{r_{12}} + \frac{e_1 e_2}{2c^2}\left[\frac{\vec{v}_1\cdot\vec{v}_2}{r_{12}} + \frac{(\vec{r}_{12}\cdot\vec{v}_1)(\vec{r}_{12}\cdot\vec{v}_2)}{r_{12}^3}\right].$$

But this is symmetrical in 1,2, so it is also the interaction Lagrangian of 1st particle in field of 2nd.

So Lagr. for 2 particles interacting with each other is

$$L = -m_1c^2\sqrt{1-\frac{v_1^2}{c^2}} - m_2c^2\sqrt{1-\frac{v_2^2}{c^2}}$$

$$-\frac{e_1 e_2}{r_{12}} + \frac{e_1 e_2}{2c^2}\left[\frac{\vec{v}_1\cdot\vec{v}_2}{r_{12}} + \frac{(\vec{r}_{12}\cdot\vec{v}_1)(\vec{r}_{12}\cdot\vec{v}_2)}{r_{12}^3}\right]$$

Might as well expand the free particle part of L thru $(v/c)^2$ terms,

$$-mc^2\sqrt{1-\frac{v^2}{c^2}} = \overset{\text{const.}}{\cancel{-mc^2}} + \frac{mv^2}{2} + \frac{mv^4}{8c^2} + \dots$$

since interaction is only computed thru $O(v/c)^2$.

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Then can generalize to collection of particles, get

$$\boxed{L = \sum_{i=1}^{n}\left(\frac{m_i}{2}v_i^2 + \frac{m_i v_i^4}{8c^2}\right) - \frac{1}{2}\sum_{i\neq j}\frac{e_i e_j}{r_{ij}} + \frac{1}{2}\sum_{i\neq j}\frac{e_i e_j}{2c^2}\left(\frac{\vec{v}_i\cdot\vec{v}_j}{r_{ij}} + \frac{(\vec{r}_{ij}\cdot\vec{v}_i)(\vec{r}_{ij}\cdot\vec{v}_j)}{r_{ij}^3}\right)}$$

Darwin Lagrangian.

(Factor of 1/2 in interaction terms to avoid double counting).

<u>Uses</u>: ① Atomic physics (He not H) for fine structure

② Condensed matter, plasma physics etc. to take into account magnetic interaction of matter but w.o. radiation.

<u>Note</u>: in bulk matter, +,− charges cancel most of electric field, leaving behind magnetic forces.

<u>Note #2:</u> The Darwin Lagrangian incorporates the EM interactions betw. particles thru. $O(v/c)^2$. The Lagr. only involves particles (no fields). It's like action-at-a-distance, but only in this approximation. The fields can be eliminated only thru $O(v/c)^2$. At $O(v/c)^3$, you get <u>radiation</u>. (There is, however, the Feynman-Wheeler theory, that eliminates the fields exactly, but requires absorbing particles at ∞.)