Summary. Wed 10/30/02 (1)

Gauge-inv. stress-energy tensor, EM field.

$$\boxed{T^{\mu\nu} = \frac{F^{\mu\alpha} F_{\alpha}{}^{\nu}}{4\pi} - g^{\mu\nu} \mathcal{L}_{em}} = T^{\nu\mu} = \underline{\text{symmetric}}$$

$$\boxed{\partial_\mu T^{\mu\nu} = -\frac{1}{c} F^{\nu}{}_{\alpha} J^{\alpha}}$$

$$T^{00} = \frac{E^2+B^2}{8\pi}, \quad T^{0i} = T^{i0} = \frac{(\vec{E}\times\vec{B})_i}{4\pi},$$

$$T^{ij} = \left(\frac{E^2+B^2}{8\pi}\right)\delta_{ij} - \frac{E_i E_j + B_i B_j}{4\pi} = -T_{ij} \text{ (Maxwell stress).}$$

$$\nu = 0: \quad \frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) + c\nabla\cdot\left(\frac{\vec{E}\times\vec{B}}{4\pi}\right) = -\vec{E}\cdot\vec{J} = -\left(\frac{\text{work}}{\text{time-vol}}\right)$$

$$\nu = j: \quad \frac{\partial}{\partial t}\left(\frac{(\vec{E}\times\vec{B})_j}{4\pi c}\right) + \partial_i T^{ij} = -\left(\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right)_j = -\left(\frac{\text{momentum}}{\text{time-vol}}\right).$$

$u$ = energy density $= \frac{E^2+B^2}{8\pi}$

$\vec{S}$ = energy flux $= \frac{c}{4\pi}\vec{E}\times\vec{B}$ = Poynting vector.

$\vec{g}$ = momentum density $= \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S}$

$T^{ij}$ = i-th comp. of flux of j-th comp. of momentum.

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RHS = $-$ (time rate of change of momentum of matter / vol.)

$= -$ ( Force / vol )

So LHS is 4-divergence of EM momentum current, i.e.

$$\boxed{\vec{g} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S} = \text{momentum density}}$$

$$\boxed{T^{ij} = \text{momentum flux.}}$$

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Next Goal: Include matter in stress-energy formalism.

1st work on a covariant expression for the current $J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$.

For single particle, know

$$\rho(\vec{x},t) = e\ \delta^3(\vec{x}-\vec{y}(t))$$

Notation:
$\vec{x}$ = field pt.
$\vec{y}$ = particle position.

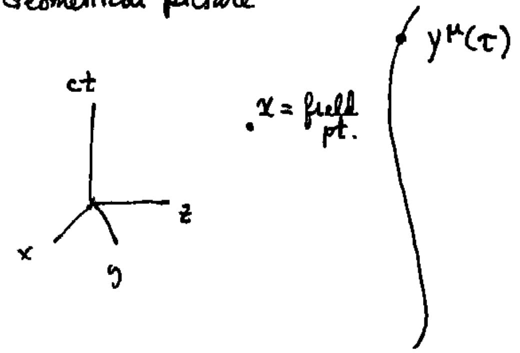
$$\vec{J}(\vec{x},t) = e\frac{d\vec{y}}{dt}\delta^3(\vec{x}-\vec{y}(t)).$$

Can write $\rho$ in another form. Let $x = (ct, \vec{x})$ = field space-time pt.

$y = (y^0, \vec{y})$ = particle space-time pt.,

$y^\mu = y^\mu(\tau)$ world line of particle.

Geometrical picture




$y^0 = ct_{particle}$

$x^0 = ct_{field}$.

$J^\mu(x) = 0$ unless $x$ lies somewhere on world line of particle.

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Turns out, you can write:

$$\rho(x) = \rho(\vec{x}, t) = e\int d\tau \; \frac{dy^0}{d\tau} \, \delta^4(x - y(\tau))$$

Proof:

$$= e\int d\tau \; \frac{dy^0}{d\tau} \, \delta(ct - y^0(\tau)) \, \delta^3(\vec{x} - \vec{y}(\tau))$$

$$= e \, \frac{dy^0}{d\tau} \, \frac{1}{\frac{dy^0}{d\tau}} \, \delta^3(\vec{x} - \vec{y}(\tau))$$ evaluated at $\tau$ - root of $ct = y^0(\tau)$.

Call this root $\tau(t)$, means $\tau$ value on particle world line corresp. to coord time $t$. At this value of $\tau$, $y^0(\tau) = ct$, and $\vec{y}(\tau) = \vec{y}(\tau(t)) \equiv \vec{y}(t)$. So... field

$$= e \, \delta^3(\vec{x} - \vec{y}(t)).$$

Similarly,

$$\vec{J}(x) = \vec{J}(\vec{x}, t) = ec\int d\tau \; \frac{d\vec{y}}{d\tau} \, \delta^4(x - y(\tau))$$

$$= ec \, \frac{d\vec{y}}{d\tau} \, \frac{1}{\frac{dy^0}{d\tau}} \, \delta^3(\vec{x} - \vec{y}(t)) = ec \, \frac{d\vec{y}}{dy^0} \, \delta^3(\vec{x} - \vec{y}(t))$$

$$= e \, \frac{d\vec{y}}{dt} \, \delta^3(\vec{x} - \vec{y}(t))$$ because $y^0 = ct$ at root.

Altogether,

$$\boxed{J^\mu(x) = \sum_n ce \int d\tau \; \frac{dy^\mu}{d\tau} \, \delta^4(x - y(\tau))}$$

This makes it obvious that $J^\mu$ is a 4-vector. by the way

where $\sum_n$ means to sum over all particles, $e, y^\mu, \tau$ really should be $e_n, y^\mu_n, \tau_n$ ($n$ = particle index).

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Now do something similar for energy and momentum of particles. First take a single particle.

$$\text{Energy} = m\gamma c^2$$

1 $$(\text{Energy density})(\vec{x},t) = m\gamma c^2 \, \delta^3(\vec{x}-\vec{y}(t))$$ as before, $\vec{x}$ = field pt, $\vec{y}$ = particle posn.

$$= mc \frac{dy^0}{d\tau} \delta^3(\vec{x}-\vec{y}(t))$$ But $c\gamma = \frac{dy^0}{d\tau}$.

$$= mc \int d\tau \, \frac{dy^0}{d\tau}\frac{dy^0}{d\tau} \, \delta^4(x-y(\tau)).$$

2 $$\text{Energy flux} = (\text{Energy density}) \times (\text{velocity})$$

$$= m\gamma c^2 \frac{d\vec{y}}{dt} \delta^3(\vec{x}-\vec{y}(t))$$

$$= mc \frac{dy^0}{d\tau}\frac{d\vec{y}}{dt} \delta^3(\vec{x}-\vec{y}(t))$$

$$= mc^2 \int d\tau \, \frac{dy^0}{d\tau}\frac{d\vec{y}}{d\tau} \, \delta^4(x-y(\tau))$$

3 $$j\text{-th comp. of momentum} = m\gamma v_j = m\gamma \frac{dy_j}{dt}.$$

$$\text{density of } j\text{-th comp. of momentum} = m\gamma \frac{dy^j}{dt} \delta^3(\vec{x}-\vec{y}(t))$$

$$= \frac{m}{c}\frac{dy^0}{d\tau}\frac{dy^j}{dt} \delta^3(\vec{x}-\vec{y}(t))$$

$$= m \int d\tau \, \frac{dy^0}{d\tau}\frac{dy^j}{d\tau} \, \delta^4(x-y(\tau)).$$

(4) i-th comp. of flux of j-th comp. of momentum

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$$= m\gamma \frac{dy^i}{dt}\frac{dy_j}{dt}\,\delta^3(\vec{x}-\vec{y}(t))$$

$$= m \frac{dy^i}{d\tau}\frac{dy_j}{dt}\,\delta^3(\vec{x}-\vec{y}(t))$$

$$= mc\int d\tau\,\frac{dy^i}{d\tau}\frac{dy^j}{d\tau}\,\delta^4(x-y(\tau)).$$

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All this makes it logical to define:

$$\boxed{T^{\mu\nu}_{\text{matter}} = \sum_n mc\int d\tau\,\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau}\,\delta^4(x-y(\tau)).}$$

sum over all particle, $m, \tau, y^\mu$ mean $m_n, \tau_n, y^\mu_n$.

Now consider divergence of $T^{\mu\nu}_{\text{matter}}$:

$$\partial_\mu T^{\mu\nu}_{\text{matter}} = \sum_n mc\int d\tau\,\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau}\,\underbrace{\frac{\partial}{\partial x^\mu}\delta^4(x-y(\tau))}_{= -\frac{\partial}{\partial y^\mu}\delta^4(x-y(\tau)).}$$

But $\dfrac{dy^\mu}{d\tau}\dfrac{\partial}{\partial y^\mu} = \dfrac{d}{d\tau}$, so

$$= -\sum_n mc\int d\tau\,\frac{dy^\nu}{d\tau}\frac{d}{d\tau}\,\delta^4(x-y(\tau))$$

integrate by parts,

$$= +\sum_n mc\int d\tau\,\frac{d^2y^\nu}{d\tau^2}\,\delta^4(x-y(\tau))$$

$$\frac{d^2y^\nu}{d\tau^2} \longrightarrow \frac{e}{mc}F^\nu{}_\alpha(y)\,\frac{dy^\alpha}{d\tau}$$

But can evaluate F at x instead of y because of δ-fn.

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So, $\partial_\mu T^{\mu\nu}_{\text{matter}} = \sum_n e \underbrace{\boxed{F^{\nu}_{\cdot\alpha}(x)}}_{\text{can take out}} \int d\tau \, \frac{dy^\alpha}{d\tau} \, \delta^4(x - y(\tau))$

$= \frac{1}{c} F^{\nu}_{\cdot\alpha}(x) \, J^\alpha(x).$ (using earlier result for $J^\alpha$)

$= -\partial_\mu T^{\mu\nu}_{em}.$

So we have altogether for energy-mom. conservation,

$$\boxed{\partial_\mu \left( T^{\mu\nu}_{em} + T^{\mu\nu}_{\text{matter}} \right) = 0}$$

Can check,

$E_{part} = \int d^3\vec{x} \; T^{00}_{matt} = \sum_n mc^2\gamma$

$\vec{p}_{part} = \frac{1}{c}\int d^3\vec{x} \; T^{0j}_{matt.} = \sum_n m\vec{v}\gamma$

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Now conservation of angular momentum. In general, angular momentum is the vector of conserved qty's that result when Noether's thm is applied to a Lagrangian that is invariant under <u>rotations</u>. This leads to a vector $\vec{L}$ of conserved qtys.

In relativistic mechanics, the Lagrangians are normally invariant under Lorentz transformations,

$$x^\mu \rightarrow \Lambda^{\mu}_{\cdot\nu} \, x^\nu$$

matrix of Lorentz transformation, elsewhere denoted $L^{\mu}_{\cdot\nu}$.

Lorentz transformations include spatial rotations as a special case.

This leads to an antisymmetric tensor of conserved quantities, call it $L^{\mu\nu}$, of which the spatial parts are related to angular momentum by

$$L^{ij} = \varepsilon_{ijk} L_k .$$

The conserved quantities are the spatial integrals of the $\mu=0$ components of a current tensor:

$$L^{\alpha\beta} = -L^{\beta\alpha} = \frac{1}{c}\int d^3\vec{x} \quad M^{0\alpha\beta},$$

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The properties and definition of the ang-mom. current tensor $M^{\mu\alpha\beta}$ can be obtained from Noether's thm. We will just quote the results.

Let $T^{\mu\nu} = T^{\nu\mu}$ be the symmetric stress-energy tensor, for the em field, for the matter, or for both. Then define:

$$M^{\mu\alpha\beta} \equiv x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}.$$

Note:

1. $M^{\mu\alpha\beta} = -M^{\mu\beta\alpha}$. (antisymmetric in $\alpha\beta$).
   $\alpha\beta$ labels the conserved qty,
   $\mu$ = component of corresp. current.

2. Compute divergence:

$$\begin{aligned}\partial_\mu M^{\mu\alpha\beta} &= \delta^\alpha_\mu T^{\mu\beta} + x^\alpha \partial_\mu T^{\mu\beta} \\ &\quad - \delta^{\beta}_{\mu} T^{\mu\alpha} - x^\beta \partial_\mu T^{\mu\alpha} \\ &= (\cancel{T^{\alpha\beta}} - \cancel{T^{\beta\alpha}}) + (x^\alpha \partial_\mu T^{\mu\beta} - x^\beta \partial_\mu T^{\mu\alpha}).\end{aligned}$$

Thus,

$$\partial_\mu M^{\mu\alpha\beta}_{em} = x^\alpha \partial_\mu T^{\mu\beta}_{em} - x^\beta \partial_\mu T^{\mu\alpha}_{em}$$

$$\text{(add)} \quad \partial_\mu M^{\mu\alpha\beta}_{matt} = x^\alpha \partial_\mu T^{\mu\beta}_{matt} - x^\beta \partial_\mu T^{\mu\alpha}_{matt}.$$

$$\boxed{\partial_\mu \left(M^{\mu\alpha\beta}_{em} + M^{\mu\alpha\beta}_{matt}\right) = 0} \qquad \text{since } \partial_\mu\left(T^{\mu\beta}_{em} + T^{\mu\beta}_{matt}\right) = 0.$$

Conservation of angular momentum for matter-field system.