Mon
10/28/02

Summary: $\mathcal{L} = \mathcal{L}(\psi, \partial_\mu \psi)$

$$\Psi(x, \varepsilon) = \psi(x - \varepsilon a) = \psi(x) - \varepsilon a^\nu (\partial_\nu \psi)$$

small ε
a^ν = const. vector.

$$\mathcal{L}(\Psi, \partial_\mu \Psi) = \mathcal{L}(\psi - \varepsilon a^\nu \psi, \ \partial_\mu \psi - \varepsilon a^\nu \partial_\nu \partial_\mu \psi)$$

$$= \mathcal{L}(\psi, \partial_\mu \psi) - \varepsilon a^\nu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \psi} (\partial_\nu \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} (\partial_\nu \partial_\mu \psi) \right)}_{= \text{ 4-divergence ?}}$$

Mon 2
10/28/02

A: Yes, consider

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \partial_\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi}(\partial_\nu \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(\partial_\nu \partial_\mu \psi) = \text{expression above, if}$$

$\mathcal{L}$ has no explicit x-dep.

So it looks like a gradient. But a gradient is a divergence, since

$$\partial_\mu \left(\delta^\mu_\nu \mathcal{L} \right) = \partial_\nu \mathcal{L}.$$

This means space-time displ. is a symmetry of $\mathcal{L}$

So...

$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi}(\partial_\nu \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(\partial_\nu \partial_\mu \psi) = \partial_\mu \left(\delta^\mu_\nu \mathcal{L} \right)}$$

an identity satisfied by any $\mathcal{L}$ w/o. explicit x-dep. Now use E.O.M.'s (E-L eqns.),

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right),$$

so LHS above becomes exact divergence:

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi \right] = \partial_\mu \left(\delta^\mu_\nu \mathcal{L} \right),$$

or

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

where (raise ν index)

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi - g^{\mu\nu} \mathcal{L}}$$

Canonical stress-energy tensor.

Let a^ν = unit vector in t, x, y, z directions, you get 4 conserved 4-currents for energy, p_x, p_y, p_z.

This means:

$$T^{00} = \text{energy density}$$
$$T^{i0} = \text{i-th component of energy flux}$$
$$T^{0j} = \text{density of } p_j$$
$$T^{ij} = \text{i-th component of flux of } p_j.$$

or at least proportional to these, you have to figure out the powers of c etc.

and the energy E and momentum $\vec{p}$ of the field at fixed time t are given by the spatial integrals of the $\nu = 0$ components..

$$E = \int_{\text{all space}} d^3\vec{x}\; T^{00}(\vec{x},t)$$
$$p_i = \int_{\text{all space}} d^3\vec{x}\; T^{0i}(\vec{x},t)$$

again × some const. These are <u>conserved</u> qtys.

Practice on scalar field to see how this works out.

Field Lagrangian for EM Field

First question, what are the fields ψ (that $\mathcal{L}$ will depend on) (analogous to the q's)? Is it $\vec{E}, \vec{B}$ that is $F_{\mu\nu}$, or $\vec{A}, \Phi$ that is A_μ? Expect 2nd order pde's to come out of E-L eqns, like $F = m\ddot{q}$, so fields should be A_μ not $F_{\mu\nu}$.

So... $\psi \to A_\mu$, and

$$\mathcal{L} = \mathcal{L}(A_\mu, \partial_\nu A_\mu).$$

Now what is $\mathcal{L}$? Free field first, call it $\mathcal{L}_{em}$. Should be Lorentz scalar, should be quadratic in the fields, ~~should~~ (because Maxwell's eqns are linear). Possible Lorentz scalars,

$$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$$

$$F_{\mu\nu} H^{\mu\nu} = -4\vec{E}\cdot\vec{B} \qquad \leftarrow \text{not invariant under parity}$$

$$A_\mu A^\mu \qquad \leftarrow \text{not gauge invariant.}$$

So choose

$$\boxed{\mathcal{L}_{em} = \frac{F_{\mu\nu} F^{\nu\mu}}{16\pi} = \frac{E^2 - B^2}{8\pi}}$$

16π is conventional factor, makes energy density come out right.

Now guess $\mathcal{L}_{int}$ = interaction Lagr. density. Must involve J^μ = current due to matter, must be Lorentz scalar, only choice is $J^\mu A_\mu$, only question is what const. in front. To make EOM's come out right, you find

$$\boxed{\mathcal{L}_{em} + \mathcal{L}_{int} = \frac{F_{\mu\nu} F^{\nu\mu}}{16\pi} - \frac{1}{c} J_\mu A^\mu}$$

Can add matter Lagr. density to this.

Work out E-L eqns. Consider $F_{\mu\nu}$ an abbreviation for $\partial_\mu A_\nu - \partial_\nu A_\mu$, since A_μ are the dynamical variables that $\mathcal{L}$ depends on (analog of the q's).

Need $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = -\frac{1}{c} J^\nu$, easy.

Need $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$, harder. Do it in detail. Write $\mathcal{L}_{em} = \dfrac{F_{\alpha\beta} F^{\beta\alpha}}{16\pi}$, or

$$\mathcal{L}_{em} = \frac{1}{16\pi} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha \right)\left(\partial^\beta A^\alpha - \partial^\alpha A^\beta \right)$$

4 terms, equal in pairs.

$$= \frac{1}{8\pi} \partial_\alpha A_\beta \left(\partial^\beta A^\alpha - \partial^\alpha A^\beta \right)$$

$$= \frac{1}{8\pi} \partial_\alpha A_\beta \, g^{\beta\sigma} g^{\alpha\tau} \left(\partial_\sigma A_\tau - \partial_\tau A_\sigma \right)$$

so
$$\frac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A_\nu)} = \frac{1}{8\pi}\Big[\delta_{\alpha\mu}\delta_{\beta\nu} \, g^{\beta\sigma} g^{\alpha\tau} \left(\partial_\sigma A_\tau - \partial_\tau A_\sigma \right) + \partial_\alpha A_\beta \, g^{\beta\sigma} g^{\alpha\tau} \left(\delta_{\sigma\mu}\delta_{\tau\nu} - \delta_{\tau\mu}\delta_{\sigma\nu} \right) \Big]$$

$$= \frac{1}{8\pi}\left[\partial_\nu A_\mu - \partial_\mu A_\nu + \partial_\nu A_\mu - \partial_\mu A_\nu \right] = \frac{1}{4\pi}\left(\partial_\nu A_\mu - \partial_\mu A_\nu \right) = \frac{1}{4\pi} F_{\nu\mu}$$

$$= -\frac{1}{4\pi} F_{\mu\nu}.$$

Thus, E-L equs are

$$\boxed{\partial^\mu \left(\frac{F^{\mu\nu}}{4\pi} \right) = \frac{1}{c} J^{\nu}}$$

E-L equs. for EM field.

These are Maxwell's equs, supposing that J^μ is given. Of course, J^μ is determined by particle motion, for which there is a particle Lagrangian.

Notice, the homogeneous Maxwell's equs,

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0$$

are a consequence of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so are already built into the fact that $\mathcal{L}$ is expressed in terms of $A_\mu, \partial_\nu A_\mu$.

Now apply Noether's Thm. to $\mathcal{L}_{em}$. (space-time translation symmetry) We can't apply it to $\mathcal{L}_{em} + \mathcal{L}_{int}$, because not invariant under space-time translations unless we add terms for matter. So just do $\mathcal{L}_{em}$.

For scalar field, we found canonical stress-energy tensor,

$$T^{\mu\nu}_{can} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi - g^{\mu\nu} \mathcal{L}.$$

For EM field we have a vector A^α of fields that $\mathcal{L}_{em}$ depends on, so

$$T^{\mu\nu}_{can} = \frac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A_\alpha)} \partial^\nu A_\alpha - g^{\mu\nu} \mathcal{L}_{em} \quad \text{(just sum over } \alpha\text{)}.$$

If you work this out, you find a conserved tensor, all right, but it has 2 problems:

1. Not symmetric, this affects conservation of angular momentum, more about that later
2. It's gauge-dependent.

But it's easy to construct a gauge-invariant stress-energy tensor, just follow steps of Noether's thm but use $F_{\mu\nu}$ instead of A_μ to guarantee gauge invariance. There are 2 steps.

Step ①. Find an identity satisfied by $\mathcal{L}_{em}$ due to invariance under space-time displacement $x^\nu \to x^\nu - \varepsilon a^\nu$. This is just like the case of scalar field, except we express $\mathcal{L}_{em}$ in terms of $F_{\mu\nu}$.

$$\mathcal{L}_{em} = \frac{F^{\mu\alpha} F_{\alpha\mu}}{16\pi}$$

$$\partial_\nu \mathcal{L}_{em} = \partial_\mu \left(\delta^\mu_\nu \mathcal{L}_{em}\right) = \frac{F^{\mu\alpha}(\partial_\nu F_{\alpha\mu})}{8\pi}$$

leads to canonical SE tensor, which we don't want.

Step ② Use EOMs to express everything in terms of divergences. Not E-L eqns, which involve A_μ, but Maxwell's eqns. Start with...

$$\partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} = 0$$

$$\text{so} \quad F^{\mu\alpha}\partial_\nu F_{\alpha\mu} = F^{\mu\alpha}\Big(\underbrace{\partial_\alpha F_{\nu\mu}}_{\text{1st}} + \underbrace{\partial_\mu F_{\alpha\nu}}_{\text{2nd}}\Big)$$

$$\text{2nd term} = F^{\mu\alpha}\partial_\mu F_{\alpha\nu} = F^{\alpha\mu}\partial_\alpha F_{\mu\nu} \qquad (\text{swap } \mu,\alpha)$$

$$= F^{\mu\alpha}\partial_\alpha F_{\nu\mu} = \text{1st term}$$

$$\text{so} \longrightarrow = \partial_\mu \left(\delta^\mu_\nu \mathcal{L}_{em}\right) = \frac{F^{\mu\alpha}(\partial_\mu F_{\alpha\nu})}{4\pi} = \partial_\mu\left(\frac{F^{\mu\alpha} F_{\alpha\nu}}{4\pi}\right) - \frac{(\partial_\mu F^{\mu\alpha}) F_{\alpha\nu}}{4\pi}$$

But $\partial_\mu F^{\mu\alpha} = \frac{4\pi}{c} J^\alpha$, so, (Maxwell's eqns)

$$\partial_\mu \left(\frac{F^{\mu\alpha} F_{\alpha\nu}}{4\pi} - \delta^\mu_\nu \mathcal{L}_{em}\right) = \frac{1}{c} F_{\alpha\nu} J^\alpha,$$

or

$$\boxed{\partial_\mu T^{\mu\nu}_{em} = -\frac{1}{c} F^{\nu}{}_{\alpha} J^\alpha}$$

where

$$T^{\mu\nu}_{em} = \frac{F^{\mu\alpha} F_\alpha{}^{\nu}}{4\pi} - g^{\mu\nu}\mathcal{L}_{em}$$

= electromagnetic stress-energy tensor.

This is the gauge-invariant S.E. tensor for E.M. field. If $J^\alpha = 0$ (free field), then $\partial_\mu T^{\mu\nu} = 0$, a set of 4 continuity eqns for energy ($\nu=0$) and momentum ($\nu = 1,2,3$). Write out components of $T^{\mu\nu}$ explicitly:

$$T^{\mu\nu} = \frac{1}{4\pi}\begin{bmatrix} \frac{E^2+B^2}{2} & (\vec{E}\times\vec{B})_x & (\vec{E}\times\vec{B})_y & (\vec{E}\times\vec{B})_z \\ & \frac{E^2+B^2}{2} - E_x^2 - B_x^2 & -E_xE_y - B_xB_y & -E_xE_z - B_xB_z \\ & & \frac{E^2+B^2}{2} - E_y^2 - B_y^2 & -E_yE_z - B_yB_z \\ \text{symmetric} & & & \frac{E^2+B^2}{2} - E_z^2 - B_z^2 \end{bmatrix}$$

or $T^{00} = \dfrac{E^2+B^2}{8\pi}$ $\quad T^{0i} = T^{i0} = \dfrac{(\vec{E}\times\vec{B})_i}{4\pi}$

$T^{ij} = \dfrac{E^2+B^2}{8\pi}\delta_{ij} - \dfrac{E_iE_j + B_iB_j}{4\pi}$ = negative of Maxwell stress tensor.

Look at conservation laws, use RHS to interpret physically.

$\boxed{\nu = 0:}$ $\quad \partial_\mu T^{\mu 0} = -\frac{1}{c} F^0{}_\alpha J^\alpha$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) + \nabla\cdot\left(\frac{\vec{E}\times\vec{B}}{4\pi}\right) = -\frac{1}{c}\vec{E}\cdot\vec{J}$$

mult thru. by c.

Since $\vec{E}\cdot\vec{J}$ = rate EM fields are doing work on matter /vol.,

$\Rightarrow$ $\boxed{u = \frac{E^2+B^2}{8\pi} = \text{energy density}}$

$\boxed{\vec{S} = c\,\frac{\vec{E}\times\vec{B}}{4\pi} = \text{energy flux} = \text{Poynting vector}}$

$\boxed{\nu = i}$ $\quad \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})_i + \partial_j T^{ji} = -\left(\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right)_i$