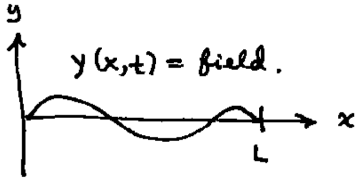
Fri 10/25/02

Summary.

1D Vibrating String.



$$A[y(x,t)] = \int dt\, L = \int dx\,dt\, \mathcal{L}. \qquad L = \int dx\, \mathcal{L}$$

$$\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{\kappa}{2}\left(\frac{\partial y}{\partial x}\right)^2.$$

Generalize: Write ψ instead of y

write $x \to \vec{x}$ (3D)

Let $\mathcal{L} = \mathcal{L}\left(\frac{\partial \psi}{\partial t}, \nabla\psi, \psi\right)$ ← depends on ψ as well as its deriv's.

$$A = \int dt\, L = \int dt\, d^3\vec{x}\, \mathcal{L} = \int d^4x\, \mathcal{L}.$$

Hamilton's principle: $\psi(\vec{x},t) \to \psi(\vec{x},t) + \delta\psi(\vec{x},t)$

$$\delta A = A[\psi(\vec{x},t) + \delta\psi(\vec{x},t)] - A[\psi(\vec{x},t)]$$

$$= \int d^4x\, \mathcal{L}\left(\psi + \delta\psi, \frac{\partial\psi}{\partial t} + \frac{\partial}{\partial t}(\delta\psi), \nabla\psi + \nabla(\delta\psi)\right) - \int d^4x\, \mathcal{L}\left(\psi, \frac{\partial\psi}{\partial t}, \nabla\psi\right) =$$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\psi} + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)}\frac{\partial}{\partial t}(\delta\psi) + \frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\cdot\nabla(\delta\psi)\right]$$

(integrate by parts)

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)}\right) - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\right)\right]\delta\psi + \text{bdry terms}$$

↑ argue away

$= 0$ demand for all $\delta\psi$ $\Rightarrow$

$$\boxed{\frac{\partial\mathcal{L}}{\partial\psi} = \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)}\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\right)}$$

3+1 version Euler-Lagrange eqns for $\mathcal{L}\left(\psi, \frac{\partial\psi}{\partial t}, \nabla\psi\right)$.

Fri. 10/25/02

Apply to vibrating string. Change notation, $\psi \to y$, $\vec{x} \to x$ (1D now).

notice $\mathcal{L}$ does not depend on y,

$$\mathcal{L} = \mathcal{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right).$$

Thus
$$0 = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial y}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial y}{\partial x}\right)}\right)$$

OR
$$0 = \frac{\partial}{\partial t}\left(\rho \frac{\partial y}{\partial t}\right) + \frac{\partial}{\partial x}\left(-\kappa \frac{\partial y}{\partial x}\right)$$

OR
$$0 = \rho \frac{\partial^2 y}{\partial t^2} - \kappa \frac{\partial^2 y}{\partial x^2}$$

or
$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$
wave equ. w. phase vel. $v = \sqrt{\frac{\kappa}{\rho}}$.

Rewrite E-L Equs in 4-vector notation. $x^\mu = (ct, \vec{x})$.

$$\mathcal{L} = \mathcal{L}(\psi, \partial_\mu \psi).$$

Then
$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right)}$$
RHS is a 4-divergence.

The field ψ and Lagr. density $\mathcal{L}$ don't have to be relativistic, but if they are it's nice that E-L equs automatically emerge in covariant form.

Now look at relativistic scalar field (good practice before tackling EM field).

Let $\psi = \psi(\vec{x}, t) = \psi(x^\mu)$. $x^\mu = (ct, \vec{x})$.

Let
$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \psi)(\partial^\mu \psi)}_{= \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial \psi}{\partial t}\right)^2 - |\nabla \psi|^2\right]} - \frac{\mu^2}{2}\psi^2.$$

just like vibrating string.

$$\mu = \frac{mc}{\hbar} = \frac{1}{\lambda_c},$$
λ_c = Compton wavelength of particle.

Work out EL equs from $\mathcal{L}$: Fri. 10/25/02

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \partial^\mu \psi , \qquad \frac{\partial \mathcal{L}}{\partial \psi} = -\mu^2 \psi .$$

So, EL equs are
$$\boxed{\begin{array}{l} \partial_\mu \partial^\mu \psi + \mu^2 \psi = 0 \\ \text{or } (\Box + \mu^2)\psi = 0 \end{array}}$$

If $\mu = 0$, it's the wave equ, as for light waves, but a scalar field.
If $\mu \neq 0$, it's the Klein-Gordon equ, the relativistic Schrödinger equ. for a relativistic, free, spinless particle.

If the photon had mass, you would have μ^2 terms in Maxwell's equs. too.

Conserved Quantities in field theory are usually associated with currents. Consider for example charge. $J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$,

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} .$$

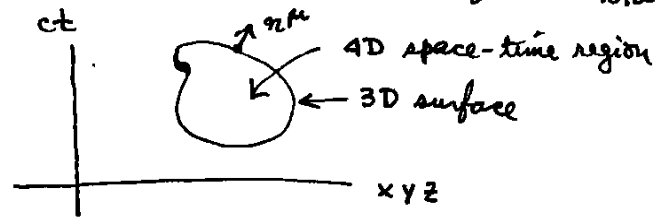
$$Q = \text{total chg at time } t = \int_{\text{all space}} d^3\vec{x} \;\; \rho(\vec{x}, t).$$

So
$$\frac{dQ}{dt} = \int d^3\vec{x} \;\; \frac{\partial \rho}{\partial t} = -\int d^3\vec{x} \;\; \nabla \cdot \vec{J} = -\int_{\text{surf.}} da \;\; \vec{J} \cdot \hat{n} = 0$$

if we assume $\vec{J} \to 0$ at spatial ∞.

Fri 10/25/02

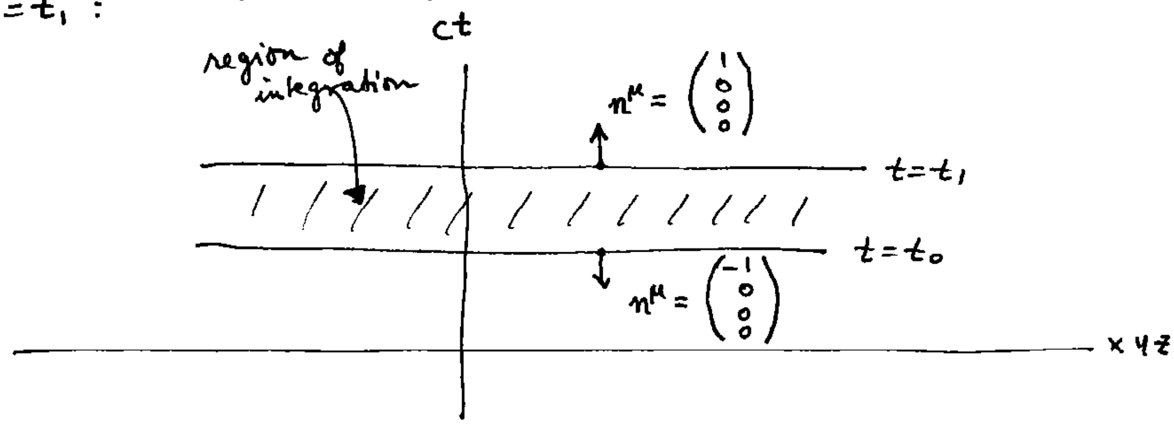
It's interesting that this can also be expressed in terms of a 4-dim'l version of Gauss' thm.



$$0 = \int_{4D\text{ region}} d^4x \, (\partial_\mu J^\mu) = \int_{\text{surf.}} d^3S \; J_\mu n^\mu$$

n^μ = unit vector normal to surface.

Now let 4D region be region between 2 const time slices, $t=t_0$ and $t=t_1$:



Then $$0 = \int_{\substack{\text{all}\\\text{space}}} d^3\vec{x} \;\; J_0(t_1, \vec{x}) - \int_{\substack{\text{all}\\\text{space}}} d^3\vec{x} \;\; J_0(t_0, \vec{x}).$$

But since $J_0 = c\rho$, this gives $Q_1 = Q_0$, charge conserved.

Fri. 10/25/02

Now Noether's Thm in Field Theory

Like $q \to Q(q,\varepsilon)$, write

$$\psi(x) \to \Psi(x,\varepsilon) \quad \text{for fields. Old field goes to new field.}$$

What is a "symmetry"? Don't demand that $\mathcal{L}$ is invariant,

$$\mathcal{L}(\Psi, \partial_\mu \Psi) = \mathcal{L}(\psi, \partial_\mu\psi) \qquad \text{(too strong).}$$

Instead demand,

$$\mathcal{L}(\Psi, \partial_\mu \Psi) = \mathcal{L}(\psi, \partial_\mu \psi) + \underbrace{(\partial_\mu G^\mu)}_{\text{a 4-divergence}}$$

like term $\frac{dG}{dt}$ for discrete Lagrangians

because this leaves EOM's invariant. This is because,

$$A = \int d^4x\, \mathcal{L} \quad \to \quad \text{itself} + \underbrace{\int d^4x\, (\partial_\mu G^\mu)}_{= \text{ bdry terms by 4D Gauss' thm}}$$

Condition $\delta A = 0$ (Ham's principle) not changed by bdry terms.

Instead of trying to derive general Noether's thm. for fields, just look at space-time translations, so

$$\Psi(x,\varepsilon) = \psi(x - \varepsilon a) = \psi(x) - \varepsilon a^\nu (\partial_\nu \psi) \qquad \text{small } \varepsilon.$$

a^ν = some const vector.

Now suppose $\mathcal{L}$ has no explicit x^μ-dependence,

$$\mathcal{L} = \mathcal{L}(\psi, \partial_\mu\psi) \quad \text{but not} \quad \mathcal{L} = \mathcal{L}(\psi, \partial_\mu\psi, x).$$

Then

$$\mathcal{L}(\Psi, \partial_\mu \Psi) = \mathcal{L}\left(\psi - \varepsilon a^\nu(\partial_\nu\psi),\ \partial_\mu\psi - \varepsilon a^\nu(\partial_\nu\partial_\mu\psi)\right)$$

$$= \mathcal{L}(\psi, \partial_\mu\psi) - \varepsilon a^\nu \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\psi}(\partial_\nu\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(\partial_\nu\partial_\mu\psi)\right)}_{}.$$

Q: Is this term ( ) a 4-divergence?