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$\boxed{\text{Now Noether's Thm.}}$

If L is independent of one q, say, q_1, then

$$\frac{dp_1}{dt} = \frac{\partial L}{\partial q_1} = 0 \quad \Rightarrow \quad p_1 = \underline{\text{constant}}.\ \underline{\text{of}}\ \underline{\text{motion}}.$$

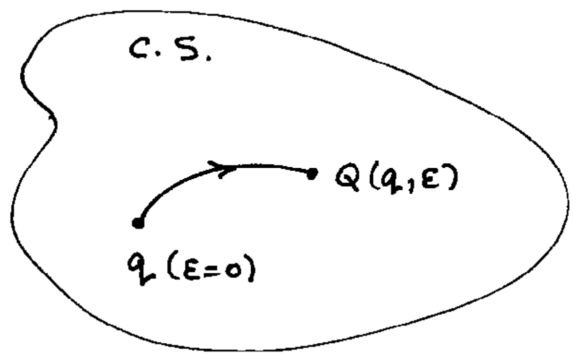
(C.O.M.)

We say, q_1 is "ignorable". Noether's thm generalizes this.

Consider a mapping of config. space onto itself parameterized by ε:

$$q \longrightarrow Q(q,\varepsilon) \quad \text{where} \quad Q(q,0) = 0.$$

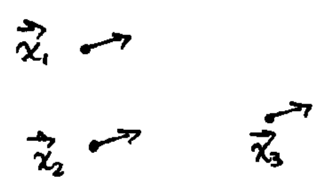
As ε goes from 0 to some final value, q gets up and traces out a curve:



<u>Example:</u> n particle system, coords $\{\vec{x}_1, \dots, \vec{x}_n\}$.

Let these be mapped into $\{\vec{x}_1 + \varepsilon\hat{a}, \vec{x}_2 + \varepsilon\hat{a}, \dots, \vec{x}_n + \varepsilon\hat{a}\}$

$\hat{a}$ = some unit vector. All particles move by same displacement, $\varepsilon\hat{a}$.



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Now *suppose* Lagrangian is *invariant* under this change,

$$L\left(Q(q,\varepsilon), \underbrace{\frac{d}{dt} Q(q,\varepsilon)}_{\text{means, } \frac{\partial Q}{\partial q}\dot{q}}, t \right) = L(q, \dot{q}, t) \quad \text{for all } \varepsilon.$$

Then we say, $q \longrightarrow Q(q,\varepsilon)$ is a *symmetry* of the system.

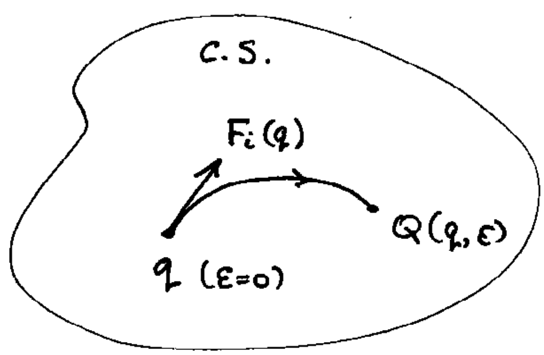
Turns out you only need to look at this for small ε, so expand $Q(q,\varepsilon)$ about $\varepsilon = 0$:

$$Q_i(q,\varepsilon) = q_i + \varepsilon F_i(q) + \dots \qquad \text{where } F_i(q) = \frac{\partial Q}{\partial \varepsilon}(q,0)$$

so symmetry is $q_i \longrightarrow q_i + \varepsilon F_i(q)$.

infinitesimal, now

F_i is vector giving direction and rate q moves for small ε. It is a vector field on C.S.



Many authors would write $\varepsilon F_i = \delta q_i$, meaning change in q_i under a small symmetry operation. (But don't confuse with δq that occurs in Hamilton's principle.)

Look at invariance of L for small ε:

$$L\left(q + \varepsilon F, \frac{dq}{dt} + \varepsilon \frac{dF}{dt}, t \right) = L(q, \dot{q}, t) \qquad \text{(we assume).}$$

$$\Rightarrow \varepsilon \left(\frac{\partial L}{\partial q_i} F_i + \frac{\partial L}{\partial \dot{q}_i} \frac{dF_i}{dt} \right) = 0.$$

← can also get this by setting

$$\frac{\partial}{\partial \varepsilon} L\left(Q(q,\varepsilon), \frac{d}{dt} Q(q,\varepsilon), t \right) \Big|_{\varepsilon=0} = 0.$$

This is just a property of L. But now we invoke the E.L. eqns,

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \quad \text{so}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) F_i + \frac{\partial L}{\partial \dot{q}_i}\frac{dF_i}{dt} = 0$$

or $$\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i} F_i\right] = 0}$$ Noether's thm. (Version #1).

or $$\boxed{\frac{d}{dt}\left(p_i F_i\right) = 0}$$

Example: suppose $L = L(\vec{x}_1, \ldots, \vec{x}_n; \dot{\vec{x}}_1, \ldots, \dot{\vec{x}}_n)$ is invariant under $\left.\begin{array}{l}\vec{x}_i \to \vec{x}_i + \varepsilon\hat{a} \\ \dot{\vec{x}}_i \to \dot{\vec{x}}_i\end{array}\right\}$.

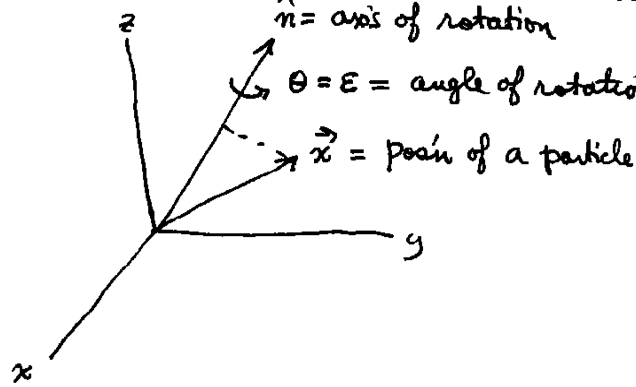
Then $\frac{d}{dt}\sum_{i=1}^{n} \vec{p}_i \cdot \hat{a} = 0$ or $\hat{a}\cdot\vec{P} = \text{C.O.M.}$

where $\vec{P} = \sum_{i=1}^{n}\vec{p}_i$ = total momentum.

If this is true for all $\hat{a}$, then $\vec{P} = \text{C.O.M.}$

Important fact: Conservation of momentum is associated with translational invariance.

Another example:



Then
$$\left.\begin{aligned} \vec{x}_i &\to \vec{x}_i + \theta\,\hat{n}\times\vec{x}_i \\ \dot{\vec{x}}_i &\to \dot{\vec{x}}_i + \theta\,\hat{n}\times\dot{\vec{x}}_i \end{aligned}\right\}$$
suppose L invariant under rotations about axis $\hat{n}$.

Then
$$\sum_{i=1}^{n} \vec{p}_i\cdot(\hat{n}\times\vec{x}_i) = \hat{n}\cdot\sum_{i=1}^{n}\vec{x}_i\times\vec{p}_i = \hat{n}\cdot\vec{L} = \text{C.O.M.}$$

If true for all $\hat{n}$, then $\vec{L}$ = C.O.M.

Conservation of angular momentum is associated with rotational invariance.

This version of Noether's thm not general enough for our needs. Turns out it's too strong to demand L be invariant under a symmetry; instead we should demand only that EOM's remain invariant. Since L and $L + \frac{d\Lambda}{dt}$, $\Lambda = \Lambda(q,t)$, give same EOM's, let us require that

$$L\left(Q(q,\varepsilon), \frac{d}{dt}Q(q,\varepsilon), t\right) = L\left(q, \frac{dq}{dt}, t\right) + \frac{d}{dt}\Lambda(q,t,\varepsilon)$$

(We might say, Lagrangian is preserved by symmetry to within a gauge transformation.). Taking $\frac{\partial}{\partial\varepsilon}$ of above and setting $\varepsilon=0$, we get

$$\left.\frac{\partial L}{\partial\varepsilon}\right|_{\varepsilon=0} = \frac{\partial L}{\partial q_i}F_i + \frac{\partial L}{\partial\dot{q}_i}\frac{d}{dt}F_i = \frac{d}{dt}G(q,t),$$
(assume this is true)

where $G(q,t) = \frac{\partial\Lambda}{\partial\varepsilon}(q,t,0)$

Now using E-L eqns gives

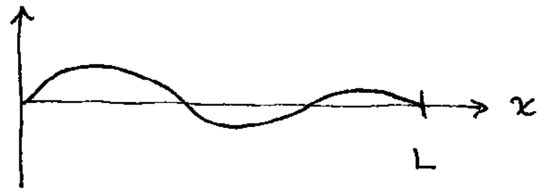
$$\boxed{\frac{d}{dt}\left(p_i F_i - G \right) = 0}$$

(Generalized version of Noether's thm.)

Put Noether's thm. on hold, turn to <u>classical</u> <u>field</u> <u>theory</u>.

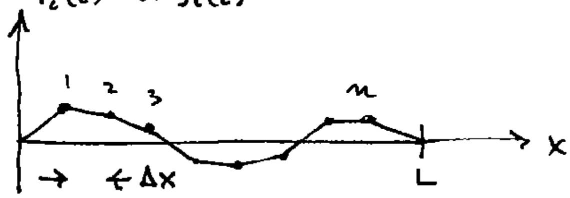
Start with vibrating string.

$\psi(x,t)$ or $y(x,t)$ = vertical posn of string.



$\psi_i(t)$ or $y_i(t)$

discretize this:



Assume masses can move up and down, not right and left. (Transverse waves only, no longitudinal waves.)

$$K.E. = \sum_{i=1}^{n} \frac{m}{2}\left(\frac{dy_i}{dt}\right)^2 .$$

m = mass of segment Δx
$= \rho \, \Delta x$

$\Delta x = x_i - x_{i-1}$

$$P.E. = \sum_{i=1}^{n+1} \frac{k}{2}\left(y_i - y_{i-1}\right)^2$$

$k = \frac{\kappa}{\Delta x}$ shorter springs are stiffer than longer springs

In limit $n \to \infty$, $y_i(t) \to y(x_i, t)$

So

$$K.E. = \sum_{i=1}^{n} \frac{1}{2}\rho \Delta x \left(\frac{dy_i}{dt}\right)^2 \to \int dx \, \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2$$

$$P.E. = \sum_{i=1}^{n+1} \frac{1}{2} \kappa \Delta x \left(\frac{y_i - y_{i-1}}{\Delta x}\right)^2 \to \int dx \, \frac{1}{2}\kappa\left(\frac{\partial y}{\partial x}\right)^2$$

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So Lagrangian becomes

$$L = \sum_{i=1}^{n} \left[\frac{m}{2} \dot{y}_i^2 - \frac{k}{2} (y_i - y_{i-1})^2 \right]$$

$$\to \int dx \; \mathcal{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x} \right) \qquad \text{where}$$

$$\mathcal{L} = \underline{\text{Lagrangian}}\ \underline{\text{density}} = \boxed{\frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{k}{2}\left(\frac{\partial y}{\partial x}\right)^2 = \mathcal{L}}$$

String Lagr. density

Henceforth write ψ instead of y, $\psi = \psi(x,t)$.

choose units so $\rho = k = 1$.

Don't confuse $\mathcal{L}$ = Lagrangian density with $L = \int dx \mathcal{L}$ = Lagrangian.

$$\text{Lagrangian} = L = \int dx \; \mathcal{L}, \quad \text{Action} = A[\psi(x,t)] = \int dx\, dt \; \mathcal{L}. = \int L\, dt.$$

Notice that index i of discretized string is replaced by continuous x. The x is a <u>label</u> of the degrees of freedom of the system, it is <u>not</u> a dynamical variable itself. The dynamical variable is ψ. A field like ψ has an ∞ # of degrees of freedom, one for each spatial point.

This is unlike particle mechanics, where x is the coordinate of the particle and also the degree of freedom.

The <u>configuration space</u> for the field is <u>not</u> x-space, it is the <u>function space</u> to which ψ belongs.