Fri. 10/18/02.

Question. Given some eqns of motion and some L that gives them, is L unique?

A: No, can add any total time derivative to L without affecting E.O.M.s,

$$L(q,\dot{q},t) \longrightarrow L(q,\dot{q},t) + \underbrace{\frac{dG(q,t)}{dt}}_{\frac{\partial G}{\partial t} + \frac{\partial G}{\partial q}\dot{q}}$$

Because,

$$\text{New } A[q(t)] = \text{Old } A[q(t)] + \int_{t_0}^{t_1} \frac{dG}{dt}\, dt$$

$$= \text{Old Action} + \underbrace{G(q_1,t_1) - G(q_0,t_0)}_{\text{a const for all paths in path space.}}$$

Hence new action stationary iff old action stationary.

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New Hamiltonians.

Given $L(q, \dot{q}, t)$, define

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad \text{(as before)}$$

$$H = \sum_i p_i \dot{q}_i - L .$$

By this formula, p_i and H come out as fns of $q, \dot{q}, t$. Now assume that defn of momentum,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}, t) = p_i(q, \dot{q}, t)$$

can be inverted to solve for $\dot{q}_i = \dot{q}_i(q, p, t)$. Such Lagrangians (that allow this to be done) are called regular. Then any fn. of $\dot{q}$ can be reexpressed as fns. of p. Normally we do this to H, so that $H = H(q, p, t)$. Notice that if Lagrangian is irregular, then H can still be defined as a fn. of $(q, \dot{q}, t)$, it just cannot be expressed as fn of (q, p, t).

Nonrelativistic mechanics usually leads to regular Lagrangians, but sometimes relativistic mechanics and field theory lead to irregular ones. When L is regular and H can be written as $H(q, p, t)$, then the E.O.M.'s are

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \quad \text{Hamilton's eqns.}$$

We'll mostly work with Lagrangians.

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<u>New Lagrangians for particle motion in EM fields.</u>

Problem: Given EOM's, how to find L? Start with free particle.
Use clue, <u>action</u> must be <u>Lorentz</u> <u>invariant</u>.

$$A[\vec{x}(t)] = \int L\,dt = \text{Lorentz inv.}$$

Guess, $L\,dt = k\,d\tau$ (k = const to be determined)

$$A = k\int d\tau = k\int \frac{dt}{\gamma} = k\int \sqrt{1-\frac{|\dot{\vec{x}}|^2}{c}}\,dt.$$

$$L = k\sqrt{1-\frac{|\dot{\vec{x}}|^2}{c^2}}. \qquad \vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = k\,\frac{\left(-\frac{1}{c^2}\right)\dot{\vec{x}}}{\sqrt{1-\frac{|\dot{\vec{x}}|^2}{c^2}}}.$$

If you want $\vec{p} = m\gamma\dot{\vec{x}}$, require $k = -mc^2$,

$$L = -mc^2\sqrt{1-\frac{|\dot{\vec{x}}|^2}{c^2}}.$$

not K.E., but if $(v/c) \ll 1$, $L \approx -mc^2 + \frac{m}{2}|\dot{\vec{x}}|^2$

and E-L eqns are $\boxed{\frac{d\vec{p}}{dt} = 0}$ correct.

Now how to include interaction with EM field. Let L_I = interaction Lagrangian. Again must have $L_I\,dt$ = Lorentz scalar, or γL_I = Lorentz scalar. Guess, another const.

$$\gamma L_I = k\left(A_\mu \frac{dx^\mu}{d\tau}\right), \qquad L_I = \gamma A_\mu \frac{dx^\mu}{dt}.$$

Find you need $k = -\frac{e}{c}$ (e = chg. of particle).

$$L_I = -\frac{e}{c}A_\mu\frac{dx^\mu}{dt} = -\frac{e}{c}\left(c\Phi - \dot{\vec{x}}\cdot\vec{A}\right).$$

Hence

$$\boxed{L = -mc^2\sqrt{1-\frac{|\dot{\vec{x}}|^2}{c^2}} - e\Phi + \frac{e}{c}\dot{\vec{x}}\cdot\vec{A}}$$

This is a 3+1 Lagrangian for relativistic mechanics.

Notice, gives

sometimes this part called kinetic momentum

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \underbrace{m\gamma\dot{\vec{x}}}_{} + \frac{e}{c}\vec{A} \quad = \text{canonical momentum}$$

In quantum mech., it's the canon. mom. that becomes $-i\hbar\nabla$.

This Lagrangian is regular, it gives the Hamiltonian,

$$H = \underbrace{\sqrt{m^2c^4 + c\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}}_{(\text{rest mass})c^2 \,+\, \text{kin. energy}} + \underbrace{e\Phi}_{\text{potential energy.}}$$

This can also be written

$$\left(p^\mu - \frac{e}{c}A^\mu\right)\left(p_\mu - \frac{e}{c}A_\mu\right) = m^2c^2$$

where p^μ is the canon 4-momentum, $p^\mu = \begin{pmatrix} H/c \\ \vec{p} \end{pmatrix} \leftarrow$ canon. 3-mom.

not the same as kinetic 4-mom which is

$$\begin{pmatrix} mc\gamma \\ m\vec{v}\gamma \end{pmatrix}$$

$\rightarrow$ also written p^μ elsewhere so you have to be careful with notation.

Now, $\boxed{\text{2nd approach to relativistic Lagrangian.}}$ Would like a treatment that treats $\vec{x}, t$ on an equal footing. Treat time like a 4th q. Then it can't be a parameter of the orbits any more.

Might think τ would be logical param, but $c^2 d\tau^2 = dx^\mu dx_\mu$, so τ not indep of the 4 q's $= (\vec{x}, t)$. So let the parameter be

$$\sigma = \text{arbitrary param.}$$

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Now we want...

$$A[x^\mu(\sigma)] = \int_{\sigma_1}^{\sigma_2} L \, d\sigma .$$ L= Lorentz scalar.

For free particle, one choice is

$$L = mc\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}} \;, \;\; = mc^2 \frac{d\tau}{d\sigma}, \;\; \text{so } \int L\, d\sigma = mc^2\int d\tau .$$

so

$$A[x^\mu(\sigma)] = mc\int\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}}\, d\sigma = mc\int\sqrt{dx^\mu dx_\mu}$$

$$= mc^2\int d\tau .$$

Action is indep. of parameterization σ: σ can be replaced by any monotonic fn. of itself, and we cannot expect EOM's to determine σ, Reparameterization invariance.

Look at momentum:

$$p_\mu = \frac{\partial L}{\partial\left(\frac{dx^\mu}{d\sigma}\right)} = mc\,\frac{\frac{dx_\mu}{d\sigma}}{\sqrt{\frac{dx^\nu}{d\sigma}\frac{dx_\nu}{d\sigma}}} = mc\,\frac{\frac{dx_\mu}{d\sigma}}{c\frac{d\tau}{d\sigma}} = m\frac{dx_\mu}{d\tau}.$$

Gives usual formula for 4-momentum. (good)

Now include interaction with EM field. You find you want,

$$\boxed{L\left(x^\mu, \frac{dx^\mu}{d\sigma}\right) = mc\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}} + \frac{e}{c}\frac{dx^\mu}{d\sigma}A_\mu .}$$

Covariant Lagrangian.

Notice $d\sigma$ drops out in $\int L\, d\sigma$.

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Show that this gives correct E.O.M.'s.

$$p_\mu = \frac{\partial L}{\partial\left(\frac{dx^\mu}{d\sigma}\right)} = m\frac{dx_\mu}{d\tau} + \frac{e}{c}A_\mu .$$

$$\frac{dp_\mu}{d\sigma} = \frac{\partial L}{\partial x^\mu}$$

$$\frac{d}{d\sigma}\left(m\frac{dx_\mu}{d\tau} + \frac{e}{c}A_\mu\right) = \frac{e}{c}\,\partial_\mu A_\nu \frac{dx^\nu}{d\sigma}$$

$$= m\frac{d}{d\sigma}\left(\frac{dx_\mu}{d\tau}\right) + \frac{e}{c}\,\partial_\nu A_\mu \frac{dx^\nu}{d\sigma} .$$

OR

$$m\frac{d}{d\sigma}\left(\frac{dx_\mu}{d\tau}\right) = \frac{e}{c}\underbrace{\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)}_{F_{\mu\nu}}\frac{dx^\nu}{d\sigma}$$

mult. by $\frac{d\sigma}{d\tau}$,

$$\boxed{m\frac{d^2x_\mu}{d\tau^2} = \frac{e}{c}F_{\mu\nu}\frac{dx^\nu}{d\tau}}$$

Indeed σ drops out.

This covariant Lagrangian is irregular, cannot solve for $\frac{dx^\mu}{d\sigma}$ in terms of p_μ, because σ cannot be determined. If you compute the "Hamiltonian" you find it vanishes:

$$H = p_\mu\frac{dx^\mu}{d\sigma} - L = 0 .$$

Hamilton's eqns in the usual sense don't exist.