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Particle Motion in EM Fields.

2 ways to write E.O.M's,

$$\boxed{mc\,\frac{du^\mu}{d\tau} = e\,F^{\mu}{}_{\nu}\,u^\nu}$$

or

$$\boxed{\begin{aligned} \frac{dE}{dt} &= e\,\vec{v}\cdot\vec{E} & E &= mc^2\gamma \\ \frac{d\vec{p}}{dt} &= e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) & \vec{p} &= m\vec{v}\gamma \end{aligned}}$$

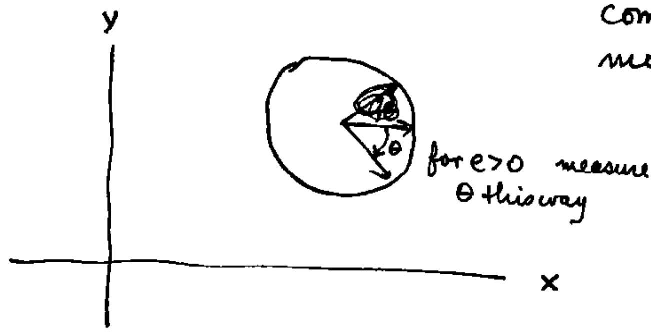
← note: $\vec{B}$ fields do no work, $\frac{dE}{dt} = \text{const.}$ in pure $\vec{B}$ field

Choose whichever is easier.

Examples:

① Const. $\vec{E}$. Easy to solve, lead to hyperbolic run away. This is the case of const. accel. in rest frame.

② Const $\vec{B}$, say $\vec{B} = B\hat{z}$. Orbit is circle in x, y plane, combined with free particle motion in z-direc'n. Hence a helix:



Facts: (a) Motion $\left\{\begin{array}{l}\text{clockwise } e>0 \\ \text{counterclockwise } e<0\end{array}\right\}$

(b) $\dfrac{d\theta}{d\tau} = \dfrac{eB}{mc}$, $\dfrac{d\theta}{dt} = \dfrac{eB}{\gamma mc} = \omega.$

(c) $E =$ const (true in any B pure field).

(d) Center of circle called "guiding center"

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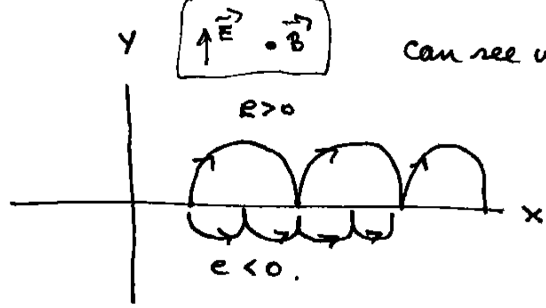
(e) a = radius, $a = \frac{v_\perp}{\omega} = \frac{mcv_\perp\gamma}{eB}$.

For other cases of const $\vec{E}, \vec{B}$, can handle by doing Lorentz transf. to frame where $\vec{E} \parallel \vec{B}$, then solve that case. "normal form" One case where this cannot be done is $\vec{E} \perp \vec{B}$, $E = B$ (both Lorentz invariants $B^2 - E^2$ and $\vec{E}\cdot\vec{B}$ vanish). But that case can be done in original frame. Leave details for exercises, except for one case to mention:

$\vec{E} \perp \vec{B}$, $E < B$.

Ex., Let $\vec{E} = E\hat{y}$, $\vec{B} = B\hat{z}$



can see what particle does.

Gives Drift velocity, $\vec{v}_D = c\,\frac{\vec{E}\times\vec{B}}{B^2}$ = avg. velocity, note $\perp$ to both $\vec{E}, \vec{B}$ (requires $E < B$).

Notice, indep. of chg e or mass m of particle. Reason is, you go to moving frame where $\vec{E} = 0$. Velocity of this frame depends only on $\vec{E}, \vec{B}$, in fact it is precisely $\vec{v}_D$. In that frame, pure $\vec{B}$ field.

Review of Lagrangian mechanics.

Many important eqns. of motion in physics (those that do not involve friction) can be written in terms of a Lagrangian plus a variational principle (Hamilton's principle). Some defns:

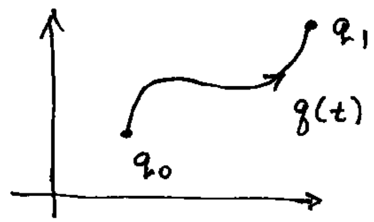
① Configuration Space = space of configurations, eg $\vec{x}_1, \ldots, \vec{x}_N$ for all particles.

② "Generalized" coordinates $(q_1, \ldots, q_n)$ = any coordinates on config. space (might be components of the $\vec{x}_i$'s).

③ n = dim (config. space) = # of degrees of freedom

④ Lagrangian = a fn. assoc. with some eqns. of motion, $L = L(q, \dot{q}, t)$.

Let q_0, q_1 be 2 pts of config. space and t_0, t_1 2 times.



⑤ A <u>path</u> = a fn $q(t)$ such that $q(t_0) = q_0$, $q(t_1) = q_1$.
It does <u>not</u> have to be a physically realizable motion.

⑥ The <u>Action</u> is a functional of paths,

$$A = A[q(t)] = \int_{t_0}^{t_1} dt \; L(q(t), \dot{q}(t), t).$$

Given $q(t)$, compute A = a number.

⑦ <u>Hamilton's</u> <u>principle</u> says that physically allowable paths $q(t)$ cause the action to be <u>stationary</u>, i.e. 1st order variations in the path give rise to only 2nd order variations in the action. (1st order variations in the action vanish).

at min, $\delta f = 0$ for arb. $\delta x, \delta y$.

$\boxed{\text{Elaborate.}}$ Let $q(t)$ = <u>some</u> <u>path</u>, $q(t) + \varepsilon(t)$ some <u>nearby</u> <u>path</u>, where $\varepsilon(t)$ is small. Note: $\varepsilon(t_0) = \varepsilon(t_1) = 0$ since

$$\left.\begin{array}{l} q(t_0) = q_0 \\ q(t_1) = q_1 \end{array}\right| \text{ and } \left.\begin{array}{l} q(t_0) + \varepsilon(t_0) = q_0 \\ q(t_1) + \varepsilon(t_1) = q_1. \end{array}\right\} \text{ path and nearby path.}$$

Here "path" means only those $q(t)$ that satisfy the end point conditions. <u>Another</u> <u>notation</u> for $\varepsilon(t)$ is $\delta q(t)$, but that is sometimes confusing. But one can write $q(t) + \delta q(t)$ for the nearby path.

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Now let

$$\delta A = A[q(t)+\varepsilon(t)] - A[q(t)]$$

$$= \int_{t_0}^{t_1} \underbrace{L(q(t)+\varepsilon(t), \dot{q}(t)+\dot{\varepsilon}(t), t)}_{\to\ L(q(t),\dot{q}(t),t) + \frac{\partial L}{\partial q}\varepsilon(t) + \frac{\partial L}{\partial \dot{q}}\dot{\varepsilon}(t)} dt - \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)\, dt$$

$$= \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q}\varepsilon(t) + \frac{\partial L}{\partial \dot{q}}\dot{\varepsilon}(t) \right]$$

↑ integrate by parts

Action is stationary along physical path, generally neither maximum or minimum

$$\delta A = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right] \varepsilon(t) + \left. \frac{\partial L}{\partial \dot{q}} \varepsilon(t) \right|_{t_0}^{t_1}$$

↑ vanishes by bdry conds on $\varepsilon(t)$.

(B) Now, if we demand $\delta A = 0$ for all $\varepsilon(t)$,

$$\Rightarrow \boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}$$ Euler-Lagrange equs.

Many important equs of physics can be put into the form of the Euler Lagrange equs, for some choice of L. Why this is so is a deep question, related to quantum mechanics and path integrals. Given some equs. of motion, we may wish to find an L. This may involve some guessing. From a purely classical standpoint, Hamilton's principle is rather mysterious.

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(9) The quantity,

$$p = \frac{\partial L}{\partial \dot q}$$

is called the canonical momentum. Thus the E-L eqns can be written,

$$\frac{dp}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}.$$

(10) If you have >1 degree of freedom, just put subscripts on q_i, p_i etc,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = \frac{\partial L}{\partial q_i}.$$

Why use Lagrangians instead of eqns of motion directly? Several answers.

(1) The action $A[q(t)]$ is a scalar. Therefore the E-L eqns are covariant under arbitrary changes of coordinates on config. space,

$$Q = Q(q,t) \qquad \text{(even t-dep. ones).}$$

Form of E-L eqns same in old + new coords,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q} \quad \Leftrightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot Q}\right) = \frac{\partial L}{\partial Q}$$

just transform L as a scalar.

(2) Lagrangians give a general relation between symmetries and constants of motion (Noether's thm). More on this later.

(3) Lagrangians lead to Hamiltonians, needed to quantize a system.

(4) Easy to build in invariance principles into a Lagrangian, eg. Lorentz invariance.