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Summary.

$$\boxed{\begin{array}{ll} e^0_0 = \gamma & e^i_0 = \gamma v_i \\ e^0_j = \gamma v_j & e^i_j = \delta_{ij} + (\gamma - 1)\dfrac{v_i v_j}{v^2} \end{array}}$$

$$\boxed{\frac{ds}{d\tau} = -(b \cdot s)\, u}$$

FW transport. (c=1)

$$b = \frac{du}{d\tau}.$$

$s^\mu =$ coords of s w.r.t. $\{e_\alpha\}$ = ~~the~~ conventional rest frame.

$s^0 = 0$

$$\boxed{\frac{ds^i}{d\tau} = -\left(\frac{de_i}{d\tau} \cdot e_k\right) s^k.} = -\Omega_{ik} s^k.$$

Note: $\Omega_{ij} = \left(\dfrac{de_i}{d\tau} \cdot e_k\right) = -\Omega_{ji} = -\left(e_i \cdot \dfrac{de_k}{d\tau}\right).$

let $\omega_i = \frac{1}{2}\varepsilon_{ijk}\Omega_{jk}$,

$$\boxed{\frac{ds^i}{d\tau} = \varepsilon_{ijk}\,\omega_j\, s^k} \quad \text{or} \quad \boxed{\frac{d\vec{s}}{dt} = \vec{\omega} \times \vec{s}}$$

Physical meaning of $\vec{\omega}$.

$$\Omega_{ik} = \frac{de^0_i}{d\tau} e^0_k - \frac{de^n_i}{d\tau} e^n_k$$

$$= \frac{d}{d\tau}\left(\gamma v_i\right)\gamma v_k - \frac{d}{d\tau}\left(\delta_{in} + (\gamma-1)\frac{v_i v_n}{v^2}\right)\left(\delta_{kn} + (\gamma-1)\frac{v_k v_n}{v^2}\right).$$

$$\frac{d}{d\tau} = \gamma\frac{d}{dt}. \qquad \frac{d\gamma}{dt} = \gamma^3(\vec{v} \cdot \vec{a}) \qquad (c=1 \text{ still})$$

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Example, 1st term = $\gamma\left[\cancelto{\text{symm.}}{\gamma^3(\vec v\cdot\vec a)}\, v_i + \gamma a_i\right]\gamma v_k.$

2nd term =

$$-\gamma\left[0 + \gamma^3(\vec v\cdot\vec a)\frac{v_i v_n}{v^2} + \frac{(\gamma-1)}{v^2}(a_i v_n + v_i a_n) - 2\frac{(\gamma-1)}{v^4} v_i v_n(\vec v\cdot\vec a)\right]$$

$$\times\left[\delta_{kn} + (\gamma-1)\frac{v_k v_n}{v^2}\right]$$

Most terms are symmetric so they cancel.

$$= \gamma^3 a_i v_k - \gamma\left[\frac{(\gamma-1)^2}{v^4}\left(a_i v_k v^2 + \cancel{v_i v_k}\, \vec v\cdot\vec a\right)\right]$$

$$= (a_i v_k)\underbrace{\left\{\gamma^3 - \frac{\gamma(\gamma-1)^2}{v^2}\right\}}.$$

$$v^2 = \frac{\gamma^2-1}{\gamma^2}.$$

$$\longrightarrow \gamma^3 - \frac{\gamma(\gamma-1)^2}{\gamma^2-1}\cdot\gamma^2$$

$$=\gamma^3\left[1-\frac{(\gamma-1)^2}{\gamma^2-1}\right] = \gamma^3\left[1-\frac{\gamma-1}{\gamma+1}\right] = \frac{2\gamma^3}{\gamma+1}.$$

$$\Omega_{ij} = \frac{2\gamma^3}{\gamma+1}a_i v_k$$

$$\boxed{\vec\omega_T = \frac{\gamma^3}{\gamma+1}\vec a\times\vec v}$$

Thomas's angular velocity.

c.f. Jackson (11.119), w. γ^2 instead of γ^3

<u>Comments:</u> Why do we care about this, when $\vec\omega_T$ depends on a choice of conventional frame?

A: Because the ||-transported frame, although "rotationless" still suffers a rotation nevertheless, in a certain sense.

How Thomas precession is used in atomic physics. Wed 10/16/02

Let $\Phi(r)$ = potential of nucleus and possibly other electrons.
assume fn. of r (central force)

$q(\text{proton}) = +e$
$q(\text{electron}) = -e$ $\Phi(r) = \frac{e}{r}$ for Hydrogen.

$V(r)$ = potential in Schrödinger eqn $= -e\Phi(r)$, $\vec{E} = -\nabla\Phi = -\frac{1}{r}\frac{d\Phi}{dr}\vec{r}$.

Classical picture of electron in orbit.

In electron rest frame, there is a magnetic field:

$$\vec{B}' = -\gamma\,\frac{\vec{v}}{c}\times\vec{E}.$$

Set $\gamma \approx 1$, velocity here satisfies $\frac{v}{c} \sim \alpha \approx \frac{1}{137}$.

Then

$$\vec{B}' = +\frac{1}{c}\frac{1}{r}\frac{d\Phi}{dr}\,\vec{v}\times\vec{r}$$

$$= -\frac{1}{mc}\frac{1}{r}\frac{d\Phi}{dr}\,\vec{r}\times\vec{p} = \frac{1}{emc}\frac{1}{r}\frac{dV}{dr}\,\vec{L}.$$

Spin has interaction with $\vec{B}'$ field,

$$H = -\vec{\mu}\cdot\vec{B} \qquad \text{where} \qquad \vec{\mu} = \frac{-eg}{2mc}\vec{S}, \quad g \approx 2.$$

$$= +\frac{e}{mc}\,\vec{S}\cdot\vec{B}$$

$$\boxed{H_{so} = \frac{1}{m^2c^2}\frac{1}{r}\frac{dV}{dr}\,\vec{L}\cdot\vec{S}}$$

Unfortunately, this gives the wrong answer (×2 too large) for SO splitting).

Note, if $H = \vec{\Omega}\cdot\vec{S}$ $\vec{\Omega}$ = some vector, then

$$\dot{\vec{S}} = \frac{1}{i\hbar}[\vec{S}, H] = \vec{\Omega}\times\vec{S}.$$ So $\vec{\Omega}$ = angular velocity, of spin precession

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here predict

$$\vec{\Omega} = \frac{1}{m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L}\cdot\vec{S}.$$

Now look at Thomas angular velocity,

$$\vec{\omega}_T = \frac{\gamma^3}{1+\gamma}\frac{\vec{a}\times\vec{v}}{c^2} \approx \frac{1}{2}\frac{\vec{a}\times\vec{v}}{c^2}. \qquad (\gamma\approx 1).$$

Here $\vec{a} = -\frac{e}{m}\vec{E} = +\frac{e}{mr}\frac{d\Phi}{dr}\vec{r} = -\frac{1}{m}\frac{1}{r}\frac{dV}{dr}\vec{r}.$

So $\frac{1}{2}\frac{\vec{a}\times\vec{v}}{c^2} = -\frac{1}{2mc^2}\frac{1}{r}\frac{dV}{dr}\vec{r}\times\vec{v}$

$$\vec{\omega}_T = -\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L}.$$

So, $\vec{\omega}_T$ cancels out $\frac{1}{2}$ of $\vec{\Omega}$, gives effective H_{SO} Hamiltonian,

$$\boxed{H_{SO} = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L}\cdot\vec{S}}$$ agrees w. expt.

Now the BMT eqn. Spins of particles are like gyroscopes on spaceships, except that it's hard to accelerate particles w/o EM fields, and these have an effect (torque) on spins (magnetic fields do).

So a spin of an accelerated particle is like a gyroscope of a spaceship with extra (nonzero) torque on it.

So first look at evolution of spin of particle that is <u>not</u> accelerated.

Go into rest frame. There is some $\vec{E}, \vec{B}$ there. Hamiltonian is

$$H = -\vec{\mu}\cdot\vec{B}, \qquad \frac{gq}{2mc}\vec{S} = \vec{\mu}$$

q = charge
g = g-factor.

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Heisenberg E.O.M.,

$$\dot{\vec{S}} = \frac{gq}{2mc} \vec{S} \times \vec{B}.$$

Given that $\vec{S}$ is the 3-vector part of a 4-vector S^μ with $S^0 = 0$ in rest frame, eqns of motion can be written,

$$\underset{\substack{\uparrow \\ S^\mu}}{\frac{d}{dt}\begin{pmatrix} 0 \\ S^1 \\ S^2 \\ S^3 \end{pmatrix}} = \frac{gq}{2mc} \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_y & -B_z \\ 0 & -B_y & 0 & B_x \\ 0 & B_z & -B_x & 0 \end{pmatrix}}_{\substack{\text{looks like } F^{\mu}{}_{\nu}, \\ \text{but with } \vec{E} \text{ field zeroed out.}}} \underset{\substack{\uparrow \\ S^\mu}}{\begin{pmatrix} 0 \\ S^1 \\ S^2 \\ S^3 \end{pmatrix}}.$$

So how to we create such a tensor? Use $u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ to make projections.

Let $$\tilde{F}^{\mu}{}_{\nu} = F^{\mu}{}_{\nu} - \frac{1}{c^2} u^\mu u_\alpha F^{\alpha}{}_{\nu} - \frac{1}{c^2} F^{\mu}{}_{\alpha} u^\alpha u_\nu.$$

Two terms subtract off 0-th row, 0-th col. Also dt (rest frame) $= d\tau$.

So EOM's are equivalent to:

$$\frac{dS^\mu}{d\tau} = \frac{gq}{2mc}\left(F^{\mu}{}_{\nu} - \frac{1}{c^2} u^\mu u_\alpha F^{\alpha}{}_{\nu} - \underbrace{\cancel{\frac{1}{c^2} F^{\mu}{}_{\alpha} u^\alpha u_\nu}}_{=0 \text{ since } (S\cdot u)=0} \right) S^\nu.$$

Write this in index-free form,

$$\boxed{\frac{dS}{d\tau} = \frac{gq}{2mc}\left(F\cdot S - \frac{1}{c^2}(u\cdot F\cdot S)\, u \right).}$$

Covariant eqn. for spin precession.

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Now suppose particle is accelerated. Then we must add the evolution of s due to FW transport to the spin evolution. FW transport takes care of part $\parallel$ to u, while spin evolution governs part $\perp$ to u (the purely spatial part in rest frame.) Result is

$b = \frac{du}{d\tau}$

$$\boxed{\frac{ds}{d\tau} = \frac{gq}{2mc}\left(F.s - \frac{1}{c^2}(u.F.s)\,u\right) - \frac{1}{c^2}(s.b)\,u}$$

BMT equ

$u.F.s = u^\mu F_{\mu\nu} s^\nu$ etc.

FW transport.

Can check that this equation satisfies $\frac{d}{d\tau}(s.u) = 0$, so if s is purely spatial in rest frame at $\tau = 0$, remains so for later τ.

Usually acceleration of particle caused by EM fields, thus

$$b = \frac{du}{d\tau} = \frac{q}{mc} F.u,$$

$$s.b = \frac{q}{mc}(s.F.u)$$

So,

$$\boxed{\frac{ds}{d\tau} = \frac{q}{mc}\left[\frac{g}{2} F.s - \frac{1}{c^2}\left(\frac{g}{2} - 1\right)(u.F.s)\,u\right]}$$

Another version of BMT equ. Jackson 11.164.