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Summary:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\mu \\ \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0 \end{array}}$$

$$A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Lorenz gauge, $\partial^\mu A_\mu = 0$

Do gauge transformations

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Wave eqn for potentials:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu .$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

$$\hookrightarrow \partial^\nu (\partial_\mu A^\mu) = 0$$

in Lorenz gauge.

Define $\partial_\mu \partial^\mu = \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ = d'Alembertian.

Then $\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$ in Lorenz gauge.

Now the <u>dual</u> <u>tensor</u>.

Jackson writes $\mathcal{F}^{\mu\nu}$ instead of $H^{\mu\nu}$.

define

$$H^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} .$$

For example, $H^{01} = \frac{1}{2}\left(\varepsilon^{0123} F_{23} + \varepsilon^{0132} F_{32}\right)$

$$= F_{23} = -B_x .$$

altogether,

$$H^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} .$$

The Homogeneous maxwell eqns are simpler in terms of the dual tensor:

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0$$

$$\Leftrightarrow \quad \partial_\mu H^{\mu\nu} = 0$$

cf. $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\mu .$

Now invariants of the field tensor (not discussed by Jackson in Ch. 11, maybe elsewhere). Given

$$\boxed{F'^{\mu\nu} = L^{\mu}{}_{\cdot\alpha} L^{\nu}{}_{\cdot\beta} F^{\alpha\beta}.}$$

what properties of F are invariant? The invariants must be Lorentz scalars. There are two that can be constructed:

$$\left.\begin{aligned} \tfrac{1}{2} F^{\mu\nu} F_{\mu\nu} &= B^2 - E^2 \\ -\tfrac{1}{4} H^{\mu\nu} F_{\mu\nu} &= \vec{E}\cdot\vec{B} \end{aligned}\right\} .$$

These are the only 2 invariants. Applications: ① If $B > E$ (or $E > B$) in one Lorentz frame, then true in all frames.

② If $\vec{E} \perp \vec{B}$ in one Lorentz frame (ie $\vec{E}\cdot\vec{B} = 0$) then $\vec{E} \perp \vec{B}$ in all Lorentz frames.

Now a couple of simple applications.
First Doppler shift and aberration.

Consider a real, plane wave,

$$\psi(x^\mu) = \mathrm{Re}\ \psi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} .$$

Sound, EM wave etc. (not nec. a light wave).

Let ϕ = phase of wave $\equiv \omega t - \vec{k}\cdot\vec{x}$.

This is a Lorentz scalar, because counting wave fronts doesn't change on going from one Lorentz frame to another.

So write $\phi = x^\mu k_\mu$, $\quad x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \quad k^\mu = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix}$.

Then k^μ must transform as a 4-vector.

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Example, let wave be moving in x-direction, $\vec{k} = k\hat{x}$.

Then,
$$\begin{pmatrix} \omega'/c \\ k' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \omega/c \\ k \end{pmatrix}$$

where primes refer to wave as viewed from moving frame.

$$\left.\begin{aligned} \omega' &= \gamma(\omega - kv) \\ k' &= \gamma\left(k - \omega\frac{v}{c^2}\right) \end{aligned}\right\}.$$

Specialize to light wave, $\omega = ck$. Then $\omega' = ck'$ and

$$\omega' = \gamma(1-\beta)\,\omega = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega \approx (1-\beta)\omega.$$

Red shift, blue shift.
$\beta > 0$ $\beta < 0$.

Next example, let light wave be moving in y-direc., $\vec{k} = k\hat{y}$, $\omega = ck$.

$$\begin{pmatrix} \omega'/c \\ k_x' \\ k_y' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega/c \\ 0 \\ k \end{pmatrix}$$

$$\omega' = \gamma\omega \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \dots\right)\omega$$

$$\left.\begin{aligned} k_x' &= -\gamma\,\omega\frac{v}{c^2} = -\gamma\beta k \\ k_y' &= k. \end{aligned}\right\}$$

direction changes,

$$\left|\frac{k_x'}{k_y'}\right| = \tan\theta \approx \theta = \gamma\beta \approx \beta$$

small β.

Called ~~Aber~~ <u>Aberration</u>

For Earth, $v \approx 30$ km/sec. $\theta \approx 10^{-4} \approx 20''$.
$c \approx 3\times10^5$ km/sec.