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<u>Summary.</u>

Particle kinematics.

$$p^\mu = m\frac{dx^\mu}{d\tau} = \begin{pmatrix} mc\gamma \\ m\gamma\vec{v} \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} \qquad p^\mu p_\mu = m^2c^2,$$

$$\text{or} \quad E^2 = m^2c^4 + c^2|\vec{p}|^2$$

where $\left.\begin{aligned} E &= mc^2\gamma \\ \vec{p} &= m\gamma\vec{v} \end{aligned}\right\} =$ relativistic defn's of energy and momentum of a particle.

Motion in EM field:

$$mc\frac{du^\mu}{d\tau} = e\, F^\mu{}_\nu\, u^\nu \,, \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} = -F^{\nu\mu}$$

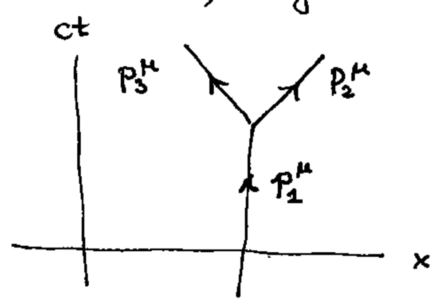
$$m\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v}\times\vec{B}) = \vec{F} = \text{force}.$$

A note about particle kinematics, collisions, decays etc.

Decay, e.g. $\pi^0 \to \gamma + \gamma$

$\pi^+ \to \mu^+ + \nu_\mu$

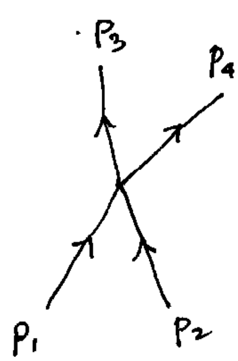
$1 \to 2 + 3$



Conservation of energy/momentum,

$$p_1^\mu = p_2^\mu + p_3^\mu.$$

or collisions $1+2 \to 3+4$



Centre of mass frame

$$P^\mu = \text{total 4-momentum} = \begin{pmatrix} E/c \\ \vec{P} \end{pmatrix}$$

COM frame is one where $\vec{P} = 0$.

A note about transformation of field tensor. Write

$$F'^{\mu\nu} = L^{\mu}_{\cdot\alpha} L^{\nu}_{\cdot\beta} F^{\alpha\beta}$$

where $\qquad x'^{\mu} = L^{\mu}_{\cdot\nu} x^{\nu}$.

Then you will find that $F'^{\mu\nu}$ contained primed $\vec{E}', \vec{B}'$ fields, same transformation law derived previously.

Now back to the transformation of the current ...

for conservation of mass.

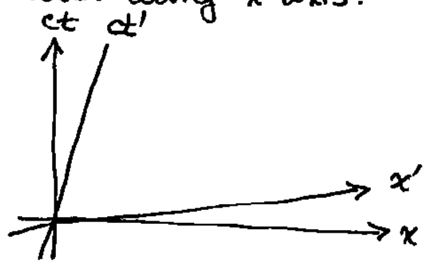
fluid mechanics. But $\begin{pmatrix} \rho_m \\ \vec{J}_m \end{pmatrix}$ does not form a 4-vector. The reason is that mass is not conserved in relativity (not rest mass, anyway), rather it is energy that is conserved, and this is not a scalar (like charge) but the 0-component of a 4-vector.

Hence Jackson's discussion of the independence of charge from the velocity: Charge is a scalar, while energy-mass is not.

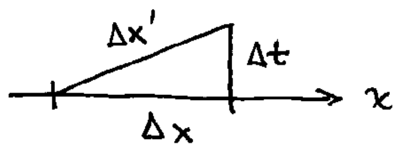
Moreover charge is conserved, hence the continuity eqn. But the continuity eqn is not needed to conclude that $(c\rho, \vec{J})$ is a 4-vector. In fact, the density, current of any scalar quantity is a 4-vector, even if it is not conserved.

The definitions of $\rho, \vec{J}$ are expressed in non-relativistic language, ρ is the charge/vol and $\vec{J}\cdot d\vec{a}$ is the charge/time crossing $d\vec{a}$. These are pre-relativistic definitions that do not change when we go to relativistic mechanics, although we can measure ρ and $\vec{J}$ (apply the definitions) in different Lorentz frames and compare the results.

So how to understand that $(c\rho, \vec{J})$ forms a 4-vector. Consider a space-time diagram for a boost along x-axis.



Look at segment of x-axis of small size Δx, and draw triangle with on side parallel to x'-axis of length $\Delta x'$ in the primed frame:



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Make a small events box $\Delta x \Delta y \Delta z$ by extending in the y, z directions. In the diagram, ~~points~~ along $\Delta x'$ segment are simultaneous in primed frame, those along Δx are simultaneous in unprimed frame. When you make a measurement of ρ, you count the charge in some spatial volume, but you have to do it at a fixed time as indicated by clocks in your reference frame. A moving observer may think he is talking about the same spatial volume, but a measurement of charge must be made at a fixed time in his frame. So the set of space-time events for the 2 measurements are not the same.

So, count charges entering or leaving the small space-time region (the triangle times $\Delta y \Delta z$ in the figure).

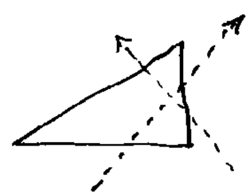
$\rho \, \Delta x \Delta y \Delta z$ = total charge whose world lines cross the x-axis ($t=0$ surface) inside volume $\Delta x \Delta y \Delta z$.

$\rho' \Delta x' \Delta y' \Delta z'$ = total charge whose world lines cross the line $\Delta x'$ ($t' =$ const surface) inside vol. $\Delta x' \Delta y' \Delta z'$.

Some charge may go out the side of the triangle:

$J_x \Delta y' \Delta z' \Delta t$ = total charge whose world lines go out the side, like this:

particle world line, must have $|v|<c$.



or charges outside may enter.

So by conservation of charge:

$$\rho \, \Delta x \Delta y \Delta z = \rho' \Delta x' \Delta y' \Delta z' + J_x \, \Delta y \Delta z \Delta t .$$

But $\Delta y' = \Delta y$
$\Delta z' = \Delta z$.

And $\Delta x' = \dfrac{\Delta x}{\gamma}$, this is the length contraction of Δx as seen by moving (primed) frame.

also, since $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$, and since $\Delta t' = 0$ along line $\Delta x'$, we have $\Delta t = \frac{v}{c^2}\Delta x$. Thus:

$$\rho\,\Delta x = \rho'\,\frac{\Delta x}{\gamma} + J_x\,\frac{v}{c^2}\,\Delta x,$$

or
$$\rho' = \gamma\left(\rho - \frac{v}{c^2}J_x\right),$$

just like the t-component of x^μ,

$$t' = \gamma\left(\rho - \frac{v}{c^2}J_x\right).$$

Similarly you find $\quad J_x' = \gamma\,(J_x - v\rho)$.

Here we have used conservation of charge but the same is true if charge is not conserved, i.e. if it satisfies

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = S = \text{source or sink}.$$

Given that $\begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix} = J^\mu =$ a 4-vector, write out the inhomogeneous Maxwell eqns in terms of $F^{\mu\nu}$ (completely covariant field tensor):

$$F^{\mu\nu} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \end{array}$$

$$\left.\begin{aligned} \nabla\cdot\vec{E} &= 4\pi\rho = \frac{4\pi}{c}J^0 \\ \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &= \frac{4\pi}{c}\vec{J} \end{aligned}\right\}\ \text{RHS} = \text{4-vector}.$$

$$\partial_1 E_x + \partial_2 E_y + \partial_3 E_z = -\partial_1 F^{01} - \partial_2 F^{02} - \partial_3 F^{03} = 4\pi\rho = \frac{4\pi}{c} J^0$$

$$\partial_2 B_z - \partial_3 B_y - \partial_0 E_x = \partial_2 F^{21} - \partial_3 F^{13} + \partial_0 F^{01} = \frac{4\pi}{c} J^1$$

$$\partial_3 B_x - \partial_1 B_z - \partial_0 E_y = \partial_3 F^{32} - \partial_1 F^{21} + \partial_0 F^{02} = \frac{4\pi}{c} J^2$$

$$\partial_1 B_y - \partial_2 B_x - \partial_0 E_z = \partial_1 F^{13} - \partial_2 F^{32} + \partial_0 F^{03} = \frac{4\pi}{c} J^3$$

or

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^{\nu}}$$

Similarly for the homog. Max. equs. you find

$$\boxed{\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0}$$

expressed in terms of covariant $F_{\mu\nu}$. Some comments:

① 4 ME's compressed into 2.

② The ~~two~~ inhomog equ. implies charge conservation, just apply ∂_ν to both sides,

$$\underbrace{\partial_\nu \partial_\mu}_{\text{symm}} \underbrace{F^{\mu\nu}}_{\text{antisymm.}} = 0 = \frac{4\pi}{c} \partial_\nu J^\nu.$$

③ If $F_{\mu\nu}$ were any antisymm. tensor, then

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu}$$

would be completely antisymmetric ~~tensor~~ (goes into − itself if any 2 indices are exchanged). But this tensor vanishes according to Maxwell.

④ Further consequences later

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Now comments about potentials. Write out $\vec{E}, \vec{B}$ in terms of potentials:

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \nabla \times \vec{A}$$

Acquires covariant form if

$$A_\mu = \begin{pmatrix} \Phi \\ -\vec{A} \end{pmatrix}, \quad A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}.$$

$$\left.\begin{aligned}
E_x &= F_{01} = -\partial_1 \Phi - \partial_0 A_x \\
E_y &= F_{02} = -\partial_2 \Phi - \partial_0 A_y \\
E_z &= F_{03} = -\partial_3 \Phi - \partial_0 A_z \\
B_x &= F_{32} = \partial_2 A_z - \partial_3 A_y \\
B_y &= F_{13} = \partial_3 A_x - \partial_1 A_z \\
B_z &= F_{21} = \partial_1 A_y - \partial_2 A_x
\end{aligned}\right\}$$

These become

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

There is a notational problem. Is A_1 the 1-covariant component of A_μ, or the x-component of $\vec{A}$? (They have opposite sign). So let A_x, A_y, A_z etc. denote components of $\vec{A}$, let A_0, A_1, A_2, A_3 denote components of A_μ.

Covariant form of gauge transformation:

Let $\vec{A} = \vec{A}' + \nabla\Lambda$ $\qquad -A_\mu = -A'_\mu + \partial_\mu \Lambda, \quad \mu = 1,2,3$

$\Phi = \Phi' - \frac{1}{c}\frac{\partial \Phi}{\partial t}$ $\qquad A_0 = A_0 - \partial_0 \Lambda, \quad \mu = 0.$

altogether,

$$\boxed{A_\mu = A'_\mu - \partial_\mu \Lambda}$$

Lorenz gauge: $\qquad L = \nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = \boxed{\partial_\mu A^\mu = 0}$ in Lorenz gauge