10/7/02

Summary

$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \qquad \begin{aligned} E_x' &= E_x \\ E_y' &= \gamma\left(E_y - \frac{v}{c}B_z\right) \\ E_z' &= \gamma\left(E_z + \frac{v}{c}B_y\right) \end{aligned} \qquad \left.\begin{aligned} B_x' &= B_x \\ B_y' &= \gamma\left(B_y + \frac{v}{c}E_z\right) \\ B_z' &= \gamma\left(B_z - \frac{v}{c}E_y\right) \end{aligned}\right\}$$

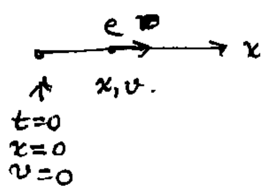
$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right)$$

$$\vec{x}\,' = \vec{x} + (\gamma - 1)\frac{(\vec{v}\cdot\vec{x})\vec{v}}{c^2} - \gamma\vec{v}t$$

1D Case:

$\vec{E} = E_x\,\hat{x}$

$\vec{B} = 0$



$$m\left.\frac{d^2x'}{dt'^2}\right|_{v'=0} = e E_x'$$

$$\boxed{m\gamma^3\frac{d^2x}{dt^2} = eE_x}$$

1

Mon 10/7/02

Now mult. both sides of this eqn. by $v = \frac{dx}{dt}$, get

$$m\gamma^3 \frac{dv}{\cancel{dt}} v = e E_x \frac{dx}{\cancel{dt}}$$

Assume electron starts at $x=0$ @ $t=0$, accelerates down x-axis.

$$m\int_0^v \frac{v\,dv}{[1-v^2/c^2]^{3/2}} = e\int_0^x E_x\,dx$$ = potential energy drop betw. 0 and x.

do integral, LHS = $mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$ = must be interpreted as kinetic energy of electron.

note, if $v/c \ll 1$, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = 1 + \frac{v^2}{2c^2}$,

Hence K.E. $= \frac{1}{2}mv^2 + \Theta\left(\frac{mv^4}{c^2}\right)$. = NR result plus H.O.T.

So we have generalized kinetic energy. Note that

$$\gamma mc^2 = mc^2 + \text{K.E.}$$

Raises suggestion that mc^2 be interpreted as "rest energy", equivalence of energy and mass. Other calculations suggest the same thing, e.g., slowly accelerated box of photons.

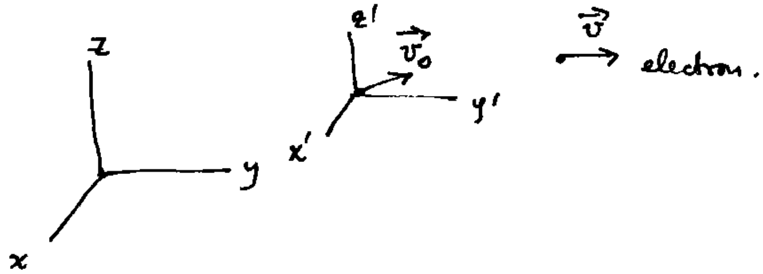
Now repeat above calculation, but do it in 3D. Just present highlights.
3D Lorentz transf:

$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right) \\ \vec{x}' &= \vec{x} + (\gamma-1)\frac{(\vec{v}\cdot\vec{x})\vec{v}}{v^2} - \gamma\vec{v}t \end{aligned}\right\}.$$

Sim. work out 3D transformation of $\vec{E}$ and $\vec{B}$ fields, by projecting $\parallel$ and $\perp$ parts w.r.t. $\vec{v}$. You find,

$$\left.\begin{aligned} \vec{E}' &= \gamma\,\vec{E} - (\gamma-1)\frac{(\vec{v}\cdot\vec{E})\vec{v}}{v^2} + \gamma\,\frac{\vec{v}}{c}\times\vec{B} \\ \vec{B}' &= \gamma\vec{B} - (\gamma-1)\frac{(\vec{v}\cdot\vec{B})\vec{v}}{v^2} - \gamma\,\frac{\vec{v}}{c}\times\vec{E} \end{aligned}\right\}$$ transformation of $\vec{E}, \vec{B}$ under general boost.

Repeat idea above. First consider electron motion from 2 frames,



Let $\left.\begin{aligned} \vec{v} &= \frac{d\vec{x}}{dt}, & \vec{v}\,' &= \frac{d\vec{x}\,'}{dt'} \\ \vec{a} &= \frac{d\vec{v}}{dt} & \vec{a}\,' &= \frac{d\vec{v}\,'}{dt'} \end{aligned}\right\}$ of electron.

$\vec{v}_0$ = frame velocity

$$\gamma_0 = \frac{1}{\sqrt{1-v_0^2/c^2}}.$$

~~then compute~~

Will assume that when $\vec{v}\,'=0$, i.e. $\vec{v}_0=\vec{v}$, then Newton's laws are valid, (in primed frame)

$$m\,\frac{d^2\vec{x}\,'}{dt'^2}\bigg|_{\vec{v}\,'=0} = m\vec{a}\,' = e\,\vec{E}'.$$

So first compute $\vec{a}\,'$ as fn. of $\vec{a}, \vec{v}$ in general, then set $\vec{v}=\vec{v}_0$.

You find,

$$\vec{a}\,' = \gamma^2\left[\vec{a} + (\gamma-1)\frac{(\vec{v}\cdot\vec{a})}{v^2}\vec{v}\right],$$ so equ of motion

in lab frame must be:

$$\boxed{m\gamma^2\left[\vec{a} + (\gamma-1)\frac{(\vec{v}\cdot\vec{a})}{v^2}\,\vec{v}\right] = e\left[\gamma\vec{E} - (\gamma-1)\frac{(\vec{v}\cdot\vec{E})\vec{v}}{v^2} + \gamma\,\frac{\vec{v}}{c}\times\vec{B}\right]}$$

③ 10/7/02

Now work on this. First dot w. $\vec{v}$: (algebra, you find)

$$\boxed{m\gamma^3 \vec{v}\cdot\vec{a} = e\,\vec{E}\cdot\vec{v}}$$

This is an energy eqn, generalization of what we had in 1D:

$$m\gamma^3\, \vec{v}\cdot d\vec{v} = e\,\vec{E}\cdot d\vec{x}$$

$$\Rightarrow mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) = e\int_{\vec{x}_0}^{\vec{x}} \vec{E}\cdot d\vec{x} \qquad \text{assuming } \vec{v} = 0 \text{ at } t = 0.$$

Again, LHS = K.E. of electron, RHS = (−pot. energy).

Use energy eqn to simplify "force" eqn : (algebra): (it simplifies)

$$\boxed{m\gamma^2\left[\vec{a} + \gamma^2\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\vec{v}\right] = e\gamma\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)}$$

Simplify further by noting,

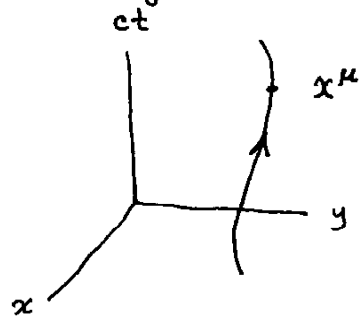
$$\frac{d\gamma}{dt} = \gamma^3\,\frac{\vec{v}\cdot\vec{a}}{c^2}$$

which makes energy and force eqn become

$$\boxed{\begin{aligned} mc^2\,\frac{d\gamma}{dt} &= e\,\vec{E}\cdot\vec{v} \\ m\,\frac{d}{dt}(\gamma\vec{v}) &= e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\end{aligned}}$$

← LHS = $\frac{d}{dt}(mc^2\gamma)$.

Now bring in ideas of 4-vectors to simplify even further. 10/7/02



2+1 space-time diagram

Recall...

$x^\mu(\tau)$ = world line parametrized by proper time

$$d\tau = \frac{dt}{\gamma}, \quad c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Let $u^\mu = \dfrac{dx^\mu}{d\tau}$ = "world velocity"

$$u^\mu = \frac{d}{d\tau}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \frac{d}{d\tau}\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} c\gamma \\ \gamma\vec{v} \end{pmatrix}.$$

Remind, $u^\mu u_\mu = c^2$.

Now, $\dfrac{dt}{d\tau} = \gamma$, so $u^0 = c\gamma$.

$$\frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \gamma\vec{v} \equiv \vec{u}$$

So energy and force eqns can be written,

$$\left.\begin{aligned} mc\,\frac{du^0}{dt} &= e\,\vec{E}\cdot\frac{d\vec{x}}{dt} \\ mc\,\frac{d\vec{u}}{dt} &= e\left(c\vec{E} + \vec{v}\times\vec{B}\right) \end{aligned}\right\}.$$

Now express everything in terms of $\frac{d}{d\tau}$ instead of $\frac{d}{dt}$, which is not invariant.

Multiply by γ:

$$mc\,\frac{du^0}{d\tau} = e\,\vec{E}\cdot\vec{u}$$

$$mc\,\frac{d\vec{u}}{d\tau} = e\left(u^0\vec{E} + \vec{u}\times\vec{B}\right)$$

or $$mc\,\frac{d}{d\tau}\begin{pmatrix}u^0\\u^1\\u^2\\u^3\end{pmatrix} = e\underbrace{\begin{bmatrix}0 & E_x & E_y & E_z\\ E_x & 0 & B_z & -B_y\\ E_y & -B_z & 0 & B_x\\ E_z & B_y & -B_x & 0\end{bmatrix}}_{\text{must be mixed tensor.}}\begin{bmatrix}u^0\\u^1\\u^2\\u^3\end{bmatrix}$$

4 vector $\frac{du^\mu}{d\tau}$ 4 vector u^μ

$$\boxed{mc\,\frac{du^\mu}{d\tau} = e\,F^\mu{}_\nu\,u^\nu}$$

Defines force law in E+M, also field tensor.

Some points to make about this result.

(1) We argued (from 1D example) that

$$mc^2\gamma = mcu^0 = \text{energy of particle.}$$

and we believe energy is conserved. But u^0 is ~~an element~~ one component of a 4-vector, and it cannot be conserved in all Lorentz frames unless the other 3 components are conserved too.

These are
$$mc\,\vec{u} = mc\gamma\,\frac{d\vec{x}}{dt} \longrightarrow c\,\vec{p} \quad \text{in N.R. limit.}$$

So let us <u>define</u> momentum $\vec{p}$ in relativistic mechanics by

$$\vec{p} = m\,\frac{d\vec{x}}{d\tau} = m\gamma\,\frac{d\vec{x}}{dt},$$

and we guess that it is conserved, too. Then

(Actually, conservation laws come from symmetries...)

$$mcu^\mu = \begin{pmatrix}E\\ c\vec{p}\end{pmatrix} \equiv c\,p^\mu.$$

4-momentum.

$$p^\mu = m\,\frac{dx^\mu}{d\tau}$$

(2) The 4-velocity u^μ has norm c^2, thus

$$u^\mu u_\mu = c^2\,\frac{d\tau^2}{d\tau^2} = c^2.$$

Hence $p^\mu p_\mu = m^2c^2 = \dfrac{E^2}{c^2} - p^2$, $\boxed{E^2 = m^2c^4 + c^2p^2}$

3 Also, since

$$u^\mu u_\mu = c^2,$$

$$\frac{du^\mu}{d\tau} u_\mu = 0.$$

The 4-acceleration must be orthogonal to the 4-velocity. Thus we must have

$$0 = mc \frac{du^\mu}{d\tau} u_\mu = e \quad u_\mu F^\mu_{\cdot\nu} u^\nu, \quad \text{or}$$

$$F_{\mu\nu} u^\mu u^\nu = 0 \quad \text{or} \quad \boxed{F_{\mu\nu} = -F_{\mu\nu}}$$

The E-M field tensor is anti-symmetric. In fact, by lowering an index,

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

Next we would like to put Maxwell's equs into covariant form. As a preliminary, we work on the current and charge density.

The continuity equ looks like a 4-dimensional divergence, i.e., if

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0 \quad \text{suggests} \quad J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$$

so that $\boxed{\partial_\mu J^\mu = 0}$. This alone strongly suggests that J^μ forms a 4-vector. But we must be careful. In N. R. mechanics we also have a continuity equ. for conservation of mass, that is, let ρ_m = mass density, $\vec{J}_m = \rho_m \vec{v}_m$ where $\vec{v}_m$ is a fluid velocity. Then $\frac{\partial \rho_m}{\partial t} + \nabla\cdot\vec{J}_m = 0$ is one of the standard equs of