

so far, we have Lorentz transformation + mathematics of tensor analysis. Fri 10/4/02

Now we need to explore electromagnetic and mechanical consequences of Lorentz transformation. We know Newtonian mechanics must be revised, we must find a new mechanics, which should be covariant under L.T.'s.

Nowadays we know that the way to do this is to express everything in terms of 4-vectors and tensors etc. But this was not so obvious in early days of relativity. Also we live in 3D not 4D, so it's important to understand relativity from a 3D (or 3+1 D) standpoint.

So we will follow many of Einstein's original arguments in the following, but gradually introduce 4-vector, 4D ideas.

To begin we need the transformations laws for \vec{E} and \vec{B} under a Lorentz transformation. Use the boost down the x-axis:

$$\left. \begin{aligned} t' &= \gamma \left(t - \frac{v}{c^2} x \right) \\ x' &= \gamma (x - vt) \\ y' &= y \\ z' &= z \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \frac{\partial}{\partial t} &= \gamma \left(\frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right) \\ \frac{\partial}{\partial x} &= \gamma \left(\frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right) \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z'} \end{aligned} \right\}.$$

Write out Maxwell eqns for \vec{E}, \vec{B} in (x,y,z,t) coords. Then transform derivatives to primed coordinates. Then search for new \vec{E}', \vec{B}' such that Maxwell eqns have the same form in new coordinates. Skip details, summarize. You find: (Use vacuum Maxwell eqns, $\rho = \vec{J} = 0$):

$$E_x' = A E_x$$

$$B_x' = A B_x$$

$$E_y' = A \gamma \left(E_y - \frac{v}{c} B_z \right)$$

$$B_y' = A \gamma \left(B_y + \frac{v}{c} E_z \right)$$

$$E_z' = A \gamma \left(E_z + \frac{v}{c} B_y \right)$$

$$B_z' = A \gamma \left(B_z - \frac{v}{c} E_y \right)$$

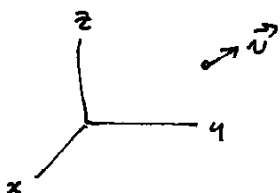
where $A = A(v)$ is some multiplier. Now invert these, to solve for \vec{E}, \vec{B} in terms of \vec{E}', \vec{B}' , and demand that inverse be given by $v \rightarrow -v$.

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Then you find $A(v)A(-v) = 1$. But $A(v)$ must $= A(-v)$, by isotropy of space, so $A^2 = 1$. But $v \rightarrow 0 \Rightarrow A = +1$, so $A = 1$. Thus we obtain the Lorentz transformed fields.

Now to generalize mechanics. We don't know what new force law will be, but we do know $F = ma$ is a good approximation at low velocities. In fact, it should be exact when $v = 0$ (it should give the exact, instantaneous acceleration, although as soon as the particle picks up speed, it will start to be only approximate.)

So suppose we have an electron moving with velocity \vec{v} at some time, in given \vec{E}, \vec{B} fields.



Then go to rest frame of electron, where $\vec{v}' = 0$, and compute accel. \vec{a}' using nonrelativistic formula,

$$m\vec{a}' = e\vec{E}' \quad (\text{no } \vec{B}' \text{ term because } \vec{v}' = 0).$$

Then use Lorentz transform to go back to lab frame, get \vec{a} in lab frame.

So we need to see how the acceleration transforms under a Lorentz transf.

For simplicity, do this in 1D (boost down x-axis).

$$\text{Let } v = \frac{dx}{dt} = \text{vel. of electron in lab. frame}$$

$$a = \frac{dv}{dt} = \text{accel. " " " " "}$$

$$v' = \frac{dx'}{dt'} = \text{vel. of electron in moving frame}$$

$$a' = \frac{dv'}{dt'} = \text{accel. " " " " "}$$

Let $v_0 =$ velocity of ^(primed) moving frame wrt lab frame.

↑ need to distinguish frame velocity from electron velocity.

Initially we don't assume $v = v_0$, i.e. $v' = 0$. γ refers to v_0 not v .

$$\gamma = \frac{1}{\sqrt{1 - v_0^2/c^2}}$$

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$$\begin{aligned} x' &= \gamma(x - v_0 t) & dx' &= \gamma(dx - v_0 dt) \\ t' &= \gamma\left(t - \frac{v_0}{c^2}x\right) & dt' &= \gamma\left(dt - \frac{v_0}{c^2}dx\right) \end{aligned}$$

$$v' = \frac{dx'}{dt'} = \frac{\gamma(dx - v_0 dt)}{\gamma\left(dt - \frac{v_0}{c^2}dx\right)} = \frac{v - v_0}{1 - \frac{v v_0}{c^2}} \quad (\text{addition of velocities}).$$

$$a' = \frac{dv'}{dt'} = \frac{\frac{dv}{1 - \frac{v v_0}{c^2}} + \frac{(v - v_0)}{\left(1 - \frac{v v_0}{c^2}\right)^2} \frac{v_0}{c^2} dv}{\gamma\left(dt - \frac{v_0}{c^2}dx\right)}$$

$$a' = \frac{a}{\gamma\left(1 - \frac{v v_0}{c^2}\right)^2} + \frac{v - v_0}{\gamma\left(1 - \frac{v v_0}{c^2}\right)^3} \frac{v_0}{c^2} a$$

Transformation law for accel. is ugly, explain why. Now we set $v_0 = v$, hence $v' = 0$. (Could not do this before differentiating.)

Also, now γ refers to v , $1 - \frac{v v_0}{c^2} = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$.

So, $a' = \gamma^3 a$. In primed frame, assume $\vec{E}' = E'_x \hat{x}$
 $\vec{B}' = 0$

$$m a' = e E'_x$$

But $E'_x = E_x$, so in lab frame,

for simplicity.
electron accelerated in
x-direction only.
 $\vec{E}' = E_x \hat{x}$
 $\vec{B}' = 0$ } $\Rightarrow E'_x = E_x$

$$\boxed{m \gamma^3 \frac{d^2 x}{dt^2} = e E_x}$$

Must be correct relativistically.