Physics 221A
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Notes 15

Irreducible Tensor Operators and the Wigner-Eckart Theorem

Our final topic in the theory of rotations and angular momentum concerns irreducible tensor operators and the Wigner-Eckart theorem. In the following discussion we will be interested in how operators transform under rotations, and how these transformation properties can be put to use in evaluating matrix elements.

First we have to define what we mean by a rotated operator. We do know how quantum states transform; if $|\psi\rangle$ is a state ket, then we define the rotated ket by

$$|\psi'\rangle = U(\mathsf{R})|\psi\rangle, \tag{15.1}$$

where $U(\mathsf{R})$ is the unitary rotation operator corresponding to a classical rotation R in the case of integral angular momenta, or one of the two unitary rotation operators, appropriately chosen, corresponding to R, in the case of half-integral angular momenta. Now we let A be an operator, and A' the rotated operator, to be defined. We make this definition by requiring the expectation value of the rotated operator with respect to the rotated state to be equal to the expectation value of the original operator with respect to the original state. In other words, we demand

$$\langle\psi|A|\psi\rangle = \langle\psi'|A'|\psi'\rangle, \tag{15.2}$$

for all states $|\psi\rangle$. But this implies

$$\langle\psi|U(\mathsf{R})^\dagger A U(\mathsf{R})|\psi\rangle = \langle\psi|A|\psi\rangle, \tag{15.3}$$

or

$$\boxed{A' = U(\mathsf{R})\, A\, U(\mathsf{R})^\dagger,} \tag{15.4}$$

which is our definition of the rotated operator.

Now it is of interest to classify operators according to their transformation properties under rotations. First we define a *scalar operator* K to be an operator which is invariant under rotations, i.e., which satisfies

$$U(\mathsf{R})\, K\, U(\mathsf{R})^\dagger = K, \tag{15.5}$$

for all operators $U(\mathsf{R})$. This terminology is obvious. Notice that it implies that a scalar operator commutes with all rotations,

$$[U(\mathsf{R}), K] = 0. \tag{15.6}$$

Next we move on to vector operators. A "vector operator" is really a vector of operators, such as the three components of the operator $\mathbf{r}$, which have certain transformation properties under rotations. The actual definition has a certain counterintuitive minus sign in it, so before presenting the definition, let us study the operator $\mathbf{r}$ which we certainly expect to be a vector operator under any reasonable definition. To be specific, we can imagine this operator acting on the state space for a spinless particle in three dimensions, in which the kets $|\mathbf{r}_0\rangle$ form a basis. Here we use the notation $\mathbf{r}_0$ for the eigenvalue (a vector of c-numbers), and the notation $\mathbf{r}$ for the operator (a vector of q-numbers). Then we study how the rotated operator acts on the basis kets. We have

$$\begin{aligned} U(\mathsf{R})\,\mathbf{r}\,U(\mathsf{R})^\dagger|\mathbf{r}_0\rangle &= U(\mathsf{R})\,\mathbf{r}|\mathsf{R}^{-1}\mathbf{r}_0\rangle = U(\mathsf{R})\big(\mathsf{R}^{-1}\mathbf{r}_0\big)|\mathsf{R}^{-1}\mathbf{r}_0\rangle \\ &= \big(\mathsf{R}^{-1}\mathbf{r}_0\big)|\mathbf{r}_0\rangle = \big(\mathsf{R}^{-1}\mathbf{r}\big)|\mathbf{r}_0\rangle, \end{aligned} \tag{15.7}$$

where in the first equality we use the definition of rotations in the theory of orbital angular momentum, Eq. (12.1), in the second we use the fact that $\mathbf{r}$ is acting on an eigenstate of itself, which brings out the eigenvalue, and in the third we use Eq. (12.1) again. But since (for the ket space in question) the states $|\mathbf{r}_0\rangle$ form a basis, we have

$$U(\mathsf{R})\,\mathbf{r}\,U(\mathsf{R})^\dagger = \mathsf{R}^{-1}\mathbf{r}. \tag{15.8}$$

This is how the archetypal vector operator $\mathbf{r}$ transforms under rotations, and we can expect other vector operators to transform similarly. The use of R^{-1} on the right hand side instead of R is somewhat counterintuitive, but it follows from the requirements we have made.

We can now define a vector operator in all generality. We say that $\mathbf{V}$ is a *vector operator* (really a vector of operators) if

$$U(\mathsf{R})\,\mathbf{V}\,U(\mathsf{R})^\dagger = \mathsf{R}^{-1}\mathbf{V}, \tag{15.9}$$

or, in components,

$$U(\mathsf{R})\,V_i\,U(\mathsf{R})^\dagger = \sum_j R_{ji}\,V_j. \tag{15.10}$$

We mention that these transformation laws can also be justified by requiring the expectation value of a vector operator to transform as a classical vector under rotations.

Given definitions (15.5) and (15.9), one can prove various theorems. For example, if $\mathbf{V}$ and $\mathbf{W}$ are vector operators, then $\mathbf{V}\cdot\mathbf{W}$ is a scalar operator, and $\mathbf{V}\times\mathbf{W}$ is a vector operator. This is of course just as in vector algebra, except that we are dealing with generally noncommutative operators here. (In particular, the order of the multiplication must be respected; for example, it is not generally true that $\mathbf{V}\cdot\mathbf{W}=\mathbf{W}\cdot\mathbf{V}$, or that $\mathbf{V}\times\mathbf{W}=-\mathbf{W}\times\mathbf{V}$.)

Finally we define a *tensor operator* T as a tensor of operators with certain transformation properties which we will illustrate in the case of a rank-2 tensor. In this case we can think of T as a matrix of operators with 9 components T_{ij}, which are required to transform according to

$$U(\mathsf{R})\,T_{ij}\,U(\mathsf{R})^\dagger=\sum_{k\ell}R_{ki}R_{\ell j}\,T_{k\ell}, \tag{15.11}$$

or, in matrix language,

$$U(\mathsf{R})\,\mathsf{T}\,U(\mathsf{R})^\dagger=\mathsf{R}^{-1}\mathsf{T}\mathsf{R}. \tag{15.12}$$

As an example of a tensor operator, let $\mathbf{V}$ and $\mathbf{W}$ be vector operators, and write

$$T_{ij}=V_iW_j. \tag{15.13}$$

Then T_{ij} is a tensor operator (it is the tensor product of $\mathbf{V}$ with $\mathbf{W}$). Tensor operators of other ranks (besides 2) are possible; a scalar is considered a tensor operator of rank 0, and a vector is considered a tensor of rank 1. In the case of tensors of arbitrary rank, the transformation law involves one copy of the matrix R^{-1} for each index of the tensor.

We notice that in all these definitions, Eqs. (15.9), (15.10), (15.11) and (15.12), we have two copies of $U(\mathsf{R})$ on the left hand side, so that in the case of half-integral angular momenta, it does not matter which of the two operators $U(\mathsf{R})$ we use to correspond to a given R, since the sign will cancel anyway. Thus, the rotated operator is determined by the classical matrix R alone.

Our definitions of scalar, vector and tensor operators are required to hold for arbitrary rotations $U(\mathsf{R})$, including infinitesimal ones. Let us therefore set

$$U(\mathsf{R})=1-\frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\mathbf{J}, \tag{15.14}$$

and

$$\mathsf{R}=\mathsf{I}+\theta\hat{\mathbf{n}}\cdot\mathbf{J}. \tag{15.15}$$

Then for scalar operators we find

$$\Big(1-\frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\mathbf{J}\Big)K\Big(1+\frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\mathbf{J}\Big)=K, \tag{15.16}$$

or,

$$[\hat{\mathbf{n}}\cdot\mathbf{J},K]=0. \tag{15.17}$$

We will also write this in the form,

$$[\mathbf{J}, K] = 0. \tag{15.18}$$

Thus, scalar operators commute with all three components of the angular momentum.

The condition (15.17) or (15.18) is not merely a consequence of the definition of a scalar operator, but actually equivalent to it. For if an operator K commutes with all three components of angular momentum, then we can trace backward to Eq. (15.16) to show that it commutes with infinitesimal rotations; and if K commutes with any two rotations, then it commutes with their product. Therefore by building up finite rotations as products of infinitesimal ones, we can show that K commutes with finite rotations. Therefore either Eq. (15.5) or (15.18) can be taken as the definition of a scalar operator.

Similarly, when we transform a vector operator under infinitesimal rotations we find,

$$\Big(1 - \frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\mathbf{J}\Big)\mathbf{V}\Big(1 + \frac{i}{\hbar}\theta\hat{\mathbf{n}}\cdot\mathbf{J}\Big) = (\mathsf{I} - \theta\hat{\mathbf{n}}\cdot\mathbf{J})\mathbf{V}, \tag{15.19}$$

or

$$[\hat{\mathbf{n}}\cdot\mathbf{J}, \mathbf{V}] = -i\hbar\,\hat{\mathbf{n}}\times\mathbf{V}. \tag{15.20}$$

We will also write this in the form,

$$[J_i, V_j] = i\hbar\,\epsilon_{ijk}\,V_k. \tag{15.21}$$

Similar commutation relations can be worked out for tensor operators of any rank.

Again, the commutation relations (15.21) can be shown to be equivalent to the definition (15.9) or (15.10), so that Eq. (15.21) can be taken as an alternative definition of a vector operator. Similar statements can be made about tensor operators of any rank.

Examples of vector operators include $\mathbf{r}$, as disscussed above, as well as the momentum $\mathbf{p}$, and therefore also the cross product $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. Here we are thinking of these operators as acting on the orbital Hilbert space of a single particle in three dimensions, for which $\mathbf{L}$ is the angular momentum. Because $\mathbf{r}$, $\mathbf{p}$ and $\mathbf{L}$ are vector operators, we have the commutation relations,

$$[L_i, r_j] = i\hbar\,\epsilon_{ijk}\,r_k, \tag{15.22a}$$

$$[L_i, p_j] = i\hbar\,\epsilon_{ijk}\,p_k, \tag{15.22b}$$

$$[L_i, L_j] = i\hbar\,\epsilon_{ijk}\,L_k. \tag{15.23c}$$

By verifying these commutation relations directly, we provide proofs that $\mathbf{r}$, $\mathbf{p}$ and $\mathbf{L}$ actually are vector operators on the ket space in question. In the case of $\mathbf{r}$, another proof, based on the alternative definition of a vector operator, Eq. (15.9), was given in Eq. (15.8). In the case of $\mathbf{L}$, the statement (15.23c) is equivalent to the usual commutation relations for angular momentum. More generally, by comparing the adjoint formula (11.57) with the commutation relations (15.21), we see that $\mathbf{J}$ is a vector operator on any Hilbert space upon which the angular momentum is defined.

We turn now to the spherical basis, which is a basis of unit vectors in ordinary 3-dimensional (physical) space which is particularly useful when studying tensor operators. This is a complex basis, even though we normally think of physical space as a real vector space. Thus, vectors which have real components with respect to the usual Cartesian basis $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ will have complex components with respect to the spherical basis.

We motivate the spherical basis by the problem of dipole radiative transitions in the hydrogen atom, in which we must evaluate matrix elements of the form

$$\langle n\ell m|\mathbf{r}|n'\ell' m'\rangle = \int d^3\mathbf{r}\, \psi_{n\ell m}(\mathbf{r})^*\, \mathbf{r}\, \psi_{n'\ell' m'}(\mathbf{r}), \tag{15.24}$$

where the energy eigenfunctions are

$$\psi_{n\ell m}(\mathbf{r}) = R_{n\ell}(r) Y_{\ell m}(\theta, \phi). \tag{15.25}$$

Here $R_{n\ell}(r)$ are the radial eigenfunctions. In evaluating such integrals, it is convenient to express the vector $\mathbf{r}$ in terms of the $Y_{\ell m}$'s for $\ell = 1$. We find

$$\begin{aligned} rY_{11}(\theta,\phi) &= -r\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{i\phi} = \sqrt{\frac{3}{4\pi}}\Big(-\frac{x+iy}{\sqrt{2}}\Big), \\ rY_{10}(\theta,\phi) &= r\sqrt{\frac{3}{4\pi}}\,\cos\theta = \sqrt{\frac{3}{4\pi}}(z), \\ rY_{1,-1}(\theta,\phi) &= r\sqrt{\frac{3}{8\pi}}\,\sin\theta e^{-i\phi} = \sqrt{\frac{3}{4\pi}}\Big(\frac{x-iy}{\sqrt{2}}\Big). \end{aligned} \tag{15.26}$$

To express these relations more compactly, we introduce the *spherical basis* given by

$$\begin{aligned} \hat{\mathbf{e}}_1 &= -\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{\sqrt{2}}, \\ \hat{\mathbf{e}}_0 &= \hat{\mathbf{z}}, \\ \hat{\mathbf{e}}_{-1} &= \frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{\sqrt{2}}, \end{aligned} \tag{15.27}$$

so that

$$rY_{1q}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\, r_q, \tag{15.28}$$

for $q = 1, 0, -1$, where

$$r_q = \hat{\mathbf{e}}_q \cdot \mathbf{r}. \tag{15.29}$$

The basis vectors $\hat{\mathbf{e}}_q$ are orthonormal in the sense that

$$\hat{\mathbf{e}}_q^* \cdot \hat{\mathbf{e}}_{q'} = \delta_{qq'}. \tag{15.30}$$

An arbitrary vector $\mathbf{X}$ can be expanded as a linear combination of the vectors $\hat{\mathbf{e}}_q^*$,

$$\mathbf{X} = \sum_q \hat{\mathbf{e}}_q^* X_q, \tag{15.31}$$

where

$$X_q = \hat{\mathbf{e}}_q \cdot \mathbf{X}. \tag{15.32}$$

These equations are equivalent to a resolution of the identity in 3-dimensional space,

$$\mathsf{I} = \sum_q \hat{\mathbf{e}}_q^* \hat{\mathbf{e}}_q, \tag{15.33}$$

in which the juxtaposition of the two vectors represents a tensor product or dyad notation.

[It is also possible to expand a vector as a linear combination of the $\hat{\mathbf{e}}_q$,

$$\mathbf{Y} = \sum_q \hat{\mathbf{e}}_q Y_q, \tag{15.34}$$

where

$$Y_q = \hat{\mathbf{e}}_q^* \cdot \mathbf{Y}. \tag{15.35}$$

These relations correspond to a different resolution of the identity,

$$\mathsf{I} = \sum_q \hat{\mathbf{e}}_q \hat{\mathbf{e}}_q^*. \tag{15.36}$$

The two types of expansion give the contravariant and covariant components of a vector; in these notes, however, we will only need the expansion indicated by Eq. (15.31).]

By means of these relations, the angular part of the integral (15.24) becomes

$$\int d\Omega\, Y_{\ell m}^*(\theta,\phi) Y_{1q}(\theta,\phi) Y_{\ell' m'}(\theta,\phi), \tag{15.37}$$

which can be evaluated by the three-$Y_{\ell m}$ formula (14.40).

The deeper reason for the interest in the spherical basis is that it actually is the standard basis in ordinary 3-dimensional space. To see this, let us note that the rotation matrices $\mathsf{R}(\hat{\mathbf{n}},\theta) = \exp(\theta\hat{\mathbf{n}}\cdot\mathbf{J})$ form a representation of the rotations acting on a vector space, namely, ordinary 3-dimensional physical space. It is what we might call the fundamental representation, since it is the one with which the theory of rotations begins. Furthermore, the matrices J satisfy the commutation relations,

$$[\mathsf{J}_i, \mathsf{J}_j] = \epsilon_{ijk}\, \mathsf{J}_k \tag{15.38}$$

[see Eq. (9.24)], and are antisymmetric. Therefore if we write

$$J_i = i\mathsf{J}_i, \tag{15.39}$$

then the matrices J_i form a vector, call it $\mathbf{J}$, of Hermitian matrices which satisfy the commutation relations,

$$[J_i, J_j] = i\epsilon_{ijk}\, J_k. \tag{15.40}$$

Furthermore, the rotation matrices now have the form,

$$\mathsf{R}(\hat{\mathbf{n}}, \theta) = \exp(-i\theta\hat{\mathbf{n}} \cdot \mathbf{J}). \tag{15.41}$$

Therefore we have a vector space (physical space) upon which a vector of Hermitian operators acts, which satisfy the standard angular momentum commutation relations, and all the conditions described at the beginning of Notes 11 are satisfied. Therefore all the conclusions of Notes 11 follow, such as the existence of a standard basis, etc. The vector space in question is not a ket space, so we will not use ket notation for the vectors of that space, but otherwise everthing else goes through.

In particular, the standard basis is a simultaneous eigenbasis of the operators J_3 and J^2. The matrices for J_i follow immediately from Eq. (9.17),

$$\begin{aligned}
J_1 = i\mathsf{J}_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
J_2 = i\mathsf{J}_2 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \\
J_3 = i\mathsf{J}_3 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{15.42}$$

from which we obtain,

$$J_\pm = \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & -i \\ \pm 1 & i & 0 \end{pmatrix} \tag{15.43}$$

and

$$J^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{15.44}$$

We see that $J^2 = 2\mathsf{I}$, which means that every vector in ordinary space is an eigenvector of J^2 with eigenvalue $j(j+1) = 2$. This means that ordinary space is an irreducible space under rotations, corresponding to $j = 1$ (which makes sense, because its dimension is $2j+1 = 3$). Therefore the standard basis consists of the eigenvectors of J_3. But we can easily check that the spherical unit vectors (15.27) are the eigenvectors of J_3, that is,

$$J_3 \hat{\mathbf{e}}_q = q \hat{\mathbf{e}}_q, \qquad q = 0, \pm 1. \tag{15.45}$$

Furthermore, the spherical unit vectors are related by the raising and lowering operators (15.43) with standard phase conventions, as one can readily check. Thus, the spherical basis vectors are vectors of a standard angular momentum basis.

It follows that the vectors of the spherical basis must transform under rotations according to Eq. (11.56), which when written in notation appropriate to the present circumstances becomes

$$\mathsf{R}\hat{\mathbf{e}}_q = \sum_{q'} \hat{\mathbf{e}}_{q'} D^1_{q'q}(\mathsf{R}). \tag{15.46}$$

An equivalent form of this relation is obtained by dotting with $\hat{\mathbf{e}}^*_{q'}$ and rearranging indices,

$$\hat{\mathbf{e}}^*_q \cdot (\mathsf{R}\hat{\mathbf{e}}_{q'}) = D^1_{qq'}(\mathsf{R}), \tag{15.47}$$

which shows that the matrices R and D^1 represent the same operator in two different bases (the Cartesian and spherical, respectively). The operator in question is the geometrical rotation operator $\mathcal{R}$ we introduced in Eq. (9.1). This concludes our discussion of the spherical basis.

Next we turn to the issue of reducibility versus irreducibility. The general notion is the following. In Notes 11, we treated the general problem of a vector space upon which a set of operators $U(\mathsf{R})$ act, representing the rotations. In those notes the vector spaces we had in mind were ket spaces representing quantum mechanical systems, but the possibility of more exotic spaces were mentioned. In any case, as discussed in those notes, we speak of a space as being *invariant* under rotations if every vector in the space is mapped into another vector of the same space under the action of every rotation operator. However, a given invariant space may possess subspaces which are also invariant under rotations; if so, we say that the given space is *reducible*. If there are no smaller invariant subspaces (apart from the trivial, null subspace), then the given space is said to be *irreducible*. If a space is reducible (irreducible) under the action of the rotation operators, then it is also reducible (irreducible) under the action of the three components of angular momentum $\mathbf{J}$, because infinitesimal rotations suffice to determine the issue of reducibility. This terminology and these basic facts can be applied to any space upon which a representation of the rotations acts.

The vector spaces we are interested in in these notes are not ket spaces, but rather vector spaces of operators. Consider, for example, the vector space of operators spanned by the three components of $\mathbf{r}$. This space consists of operators which are arbitrary linear combinations of the operators x, y and z, i.e., they are operators of the form

$$A = a_x x + a_y y + a_z z = \mathbf{a} \cdot \mathbf{r}, \tag{15.48}$$

where $\mathbf{a}$ is a vector of numbers. In this vector space of operators, the operators x, y, z serve as basis vectors (or perhaps we should say, basis operators). Thus, the operator space spanned by the three components of $\mathbf{r}$ is a 3-dimensional vector space.

The rotations act on this vector space in accordance with our definition (15.4). Furthermore, $\mathbf{r}$ is a vector operator, so we have

$$U(\mathsf{R})\, r_i \, U(\mathsf{R})^\dagger = \sum_j R_{ji}\, r_j, \tag{15.49}$$

which shows that the vector space spanned by the operators x, y and z is in fact invariant under rotations. Is this vector space of operators irreducible? The answer is yes, as can be shown without much difficulty. More generally, the 3-dimensional space of operators spanned by the three components of any vector operator is irreducible under rotations. Such a space of operators is a single irreducible space corresponding to $j = 1$ (as it must be given that it is 3-dimensional).

But the 9-dimensional space of operators spanned by the components of a rank-2 tensor operator T_{ij} is reducible. Consider, for example, the tensor operator defined by Eq. (15.13), $T_{ij} = V_i W_j$. A particular operator in the space of operators spanned by the components T_{ij} is the trace of T,

$$\operatorname{tr} \mathsf{T} = T_{11} + T_{22} + T_{33} = \mathbf{V} \cdot \mathbf{W}. \tag{15.50}$$

This is a scalar operator, and is invariant under rotations, in accordance with Eq. (15.5). Therefore by itself it forms a 1-dimensional, invariant subspace of the 9-dimensional space of operators spanned by the components of T. Therefore this 9-dimensional space is reducible. It turns out that the 9-dimensional space of operators spanned by the components of T possesses in addition two more invariant (and irreducible) subspaces. One of these is the

3-dimensional space of operators spanned by the operators,

$$\begin{aligned} X_3 &= T_{12} - T_{21} = V_1 W_2 - V_2 W_1, \\ X_1 &= T_{23} - T_{32} = V_2 W_3 - V_3 W_2, \\ X_2 &= T_{31} - T_{13} = V_3 W_1 - V_1 W_3, \end{aligned} \tag{15.51}$$

or, in other words,

$$\mathbf{X} = \mathbf{V} \times \mathbf{W}. \tag{15.52}$$

The components of $\mathbf{X}$ form a vector operator, and by themselves span an irreducible invariant subspace under rotations. The remaining irreducible subspace of operators spanned by the 9 components T_{ij} is the 5-dimensional space spanned by the following operators,

$$\begin{aligned} S_1 &= T_{12} + T_{21}, \\ S_2 &= T_{23} + T_{32}, \\ S_3 &= T_{31} + T_{13}, \\ S_4 &= T_{11} - T_{22}, \\ S_5 &= T_{11} + T_{22} - 2T_{33}. \end{aligned} \tag{15.53}$$

Thus, we say that the tensor operator $T_{ij} = V_i W_j$ is reducible; it consists of a scalar, which is the trace of T; a vector, which is effectively the antisymmetric part of T; and finally the symmetric, traceless part of T. These three parts constitute invariant irreducible subspaces of the 9-dimensional space of operators spanned by the components T_{ij}, with dimensionalities of 1, 3 and 5, respectively. You may notice that these dimensionalities are in accordance with the Clebsch-Gordan decomposition,

$$1 \otimes 1 = 0 \oplus 1 \oplus 2, \tag{15.54}$$

which corresponds to the count of dimensionalities,

$$3 \times 3 = 1 + 3 + 5 = 9. \tag{15.55}$$

This Clebsch-Gordan series arises because the vector operators $\mathbf{V}$ and $\mathbf{W}$ form two $j = 1$ irreducible subspaces of operators, and when we form T according to $T_{ij} = V_i W_j$, we are effectively combining angular momenta as indicated by Eq. (15.54). The only difference from our usual practice is that we are forming tensor products of vector spaces of operators, instead of tensor products of ket spaces.

This concludes our introduction to the notion of reducible and irreducible tensor operators, and we turn now to a special case of the Wigner-Eckart theorem, which is not only useful in its own right, but which also motivates the more general treatments which will follow.

The Wigner-Eckart theorem is useful in calculating matrix elements which commonly occur in all areas of quantum physics, especially those having to do with the emission and absorption of radiation. In addition, it is especially useful in allowing one to see quickly what the selection rules are for a given matrix element, so that one can tell when matrix elements vanish without doing any calculations at all. The Wigner-Eckart theorem is easier to remember and use than it is to prove. But you will not find the proof really difficult if you have a thorough understanding of the material that has been presented so far on rotations and angular momentum.

We begin with a special case of the Wigner-Eckart theorem, which applies to scalar operators. We consider some ket space upon which a representation of the rotations $U(\mathsf{R})$ acts, as well as their infinitesimal generators $\mathbf{J}$, which of course satisfy the standard angular momentum commutation relations. We also let K be a scalar operator acting on this same space. In practice, this ket space could be the state space for an atom, molecule, nucleus, or other system (possibly consisting of many particles, with or without spin). The following treatment is very general. The space in question and the operators acting on it satisfy the assumptions made at the beginning of Notes 11, and all the conclusions of those notes apply. In particular, a convenient basis in this ket space is the basis $|\alpha jm\rangle$, as explained in those notes.

Now consider the state $K|\alpha jm\rangle$. Since K is a scalar operator, it commutes with $\mathbf{J}$, and therefore also with J^2. Thus we easily show that

$$\begin{aligned} J^2 K|\alpha jm\rangle &= \hbar^2 j(j+1)\, K|\alpha jm\rangle, \\ J_z K|\alpha jm\rangle &= \hbar m\, K|\alpha jm\rangle, \\ J_\pm K|\alpha jm\rangle &= \hbar\sqrt{(j\mp m)(j\pm m+1)}\, K|\alpha j, m\pm 1\rangle. \end{aligned} \tag{15.56}$$

We see that $K|\alpha jm\rangle$ is a simultaneous eigenstate of J^2 and J_z corresponding to quantum numbers j and m. But this implies that $K|\alpha jm\rangle$ must be a linear combination of the states $|\alpha jm\rangle$ for the same values of j and m but possibly different values of α. That is, we must be able to write,

$$K|\alpha jm\rangle = \sum_{\alpha'} C^{jm}_{\alpha'\alpha} |\alpha' jm\rangle, \tag{15.57}$$

where the expansion coefficients are $C^{jm}_{\alpha'\alpha}$, and where the superscripts and subscripts simply indicate all the parameters upon which the expansion coefficients could depend. But actually, it turns out that the expansion coefficients do not depend on m. To show this, we apply raising or lowering operators to Eq. (15.57), finding,

$$\begin{aligned} J_\pm K|\alpha jm\rangle &= KJ_\pm|\alpha jm\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,K|\alpha j, m\pm 1\rangle \\ &= \sum_{\alpha'} C^{jm}_{\alpha'\alpha}\,J_\pm|\alpha' jm\rangle \\ &= \hbar\sqrt{(j\mp m)(j\pm m+1)}\,\sum_{\alpha'} C^{jm}_{\alpha'\alpha}\,|\alpha' j, m\pm 1\rangle, \end{aligned} \tag{15.58}$$

or

$$K|\alpha j, m\pm 1\rangle = \sum_{\alpha'} C^{jm}_{\alpha'\alpha}\,|\alpha' j, m\pm 1\rangle. \tag{15.59}$$

Comparing Eqs. (15.57) and (15.59), we see that the coefficients are independent of m, as claimed, so we write simply $C^j_{\alpha'\alpha}$ for them. Now we multiply Eq. (15.57) on the left by $\langle\alpha'' j'' m''|$ and rearrange indices, to obtain

$$\langle\alpha jm|K|\alpha' j' m'\rangle = \delta_{jj'}\,\delta_{mm'}\,C^j_{\alpha\alpha'}. \tag{15.60}$$

Thus we see that the matrix elements of a scalar operator with respect to the standard angular momentum basis are diagonal in both j and m, and they are independent of m. The first part of this conclusion, that the matrix elements are diagonal in j and m, follows more simply from the fact that K, a scalar operator, commutes with J^2 and J_z, and therefore possesses simultaneous eigenstates with these operators; but the second part of the conclusion, that the matrix elements are independent of m, is a deeper result, which depends ultimately on the rotational invariance of the scalar operator K.

An important example of a scalar operator is the Hamiltonian for an isolated system, such as an atom, molecule, or nucleus not acted upon by external forces. Such a system can consist of many particles, with or without spin; thus, the ket space is in general the tensor product of the orbital and spin ket spaces for several particles. The results above show that the standard angular momentum basis $|\alpha jm\rangle$ is a particularly convenient one in which to study the Hamiltonian for such a system, because H is already partially diagonalized in this basis. In particular, we see that the Hamiltonian is diagonal in the quantum numbers j and m, but not generally in the quantum number α. This is reasonable, since α was introduced originally as an index labeling an arbitrary set of basis kets in the (generally) degenerate simultaneous eigenspace of operators J^2 and J_z, corresponding to quantum numbers j and m. Since the choice was arbitrary, there is no reason to expect H to be diagonal with respect to α. If, however, we wish to complete the diagonalization of the Hamiltonian to find the energy eigenvalues, our work is greatly reduced by the use of the standard angular momentum basis, because we only need to perform the diagonalization in α. We see from

(15.60) that there is, in fact, one matrix in (α, α') for each value of j, but that these matrices are independent of m. Therefore the energy eigenvalues can be labeled $E_{\nu j}$, where ν is a label of the eigenvalues of the (α, α') matrix; these eigenvalues are independent of m. The corresponding eigenvectors can be labeled $|\nu jm\rangle$, and each energy level $E_{\nu j}$ is $(2j+1)$-fold degenerate (at least).

An important example of these facts which is familiar to you from undergraduate courses is central force motion in three dimensions (we neglect spin for simplicity). In this case, as discussed in Notes 12, we write $|\alpha\ell m\rangle$ for the basis kets, and we identify the index α with the label of an arbitrarily chosen basis of radial wave functions. We could set $\alpha = n$ and use some discrete basis $u_n(r)$ for the radial wavefunctions (this is what was suggested in Notes 12), or we could use the basis of wave functions $\delta(r - r_0)$, in which case α is identified with the continuous index r_0. In the latter case, the diagonalization of the Hamiltonian in (α, α') is equivalent to solving the usual radial wave equation. We recall that this radial wave equation is

$$-\frac{\hbar^2}{2\mu}\frac{d^2\chi(r)}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\chi(r) + V(r)\chi(r) = E\chi(r), \tag{15.61}$$

as we will discuss in more detail later. For now we notice that this equation is parameterized by ℓ (in the centrifugal potential), but not m, which means that the energy eigenvalues have the form $E_{n\ell}$ in general. The energy eigenkets are $|n\ell m\rangle$, and are $(2\ell+1)$-fold degenerate (at least). Here the index n corresponds to the index ν of the preceding paragraph.

The Wigner-Eckart theorem involves a generalization of these results to new classes of operators. First we make a definition. We call a set of $2k+1$ operators T^k_q, for $q = -k, \ldots, +k$, an *irreducible tensor operator* of order k, if the operators satisfy

$$\boxed{U\, T^k_q\, U^\dagger = \sum_{q'} T^k_{q'}\, D^k_{q'q}(U),} \tag{15.62}$$

for all rotation operators U. We denote the irreducible tensor operator itself by T^k, and its $2k+1$ components by T^k_q. This definition should be compared to Eq. (11.56); the point of this definition is that the components of an irreducible tensor operator transform under rotations just like the standard angular momentum basis vectors of an irreducible subspace. Thus, the order k of an irreducible tensor operator behaves like an angular momentum quantum number j, and q behaves like m. We see that the components T^k_q of an irreducible tensor operator form a standard basis in an irreducible vector space of operators.

However, unlike the standard angular momentum basis vectors, irreducible tensor operators are restricted to integer values of angular momentum quantum number k. The physical reason for this is that operators, which represent physically observable quantities, must be invariant under a rotation of 2π; the mathematical reason is that our definition of a rotated operator, given by Eq. (15.4), is quadratic $U(\mathsf{R})$, so that the representation of rotations on a vector space of operators is always a single-valued representation of $SO(3)$.

Let us examine some examples of irreducible tensor operators. A scalar operator K is an irreducible tensor operator of order 0. This follows easily from the fact that K commutes with any rotation operator U, and from the fact that the $j = 0$ rotation matrices are simply given by the 1×1 matrix (1) [see Eq. (11.51)].

Irreducible operators of order 1 are constructed from vector operators by transforming from the Cartesian basis to the spherical basis. If we let $\mathbf{V}$ be a vector operator as defined by Eq. (15.9), and define its spherical components by

$$V_q = T^1_q = \hat{\mathbf{e}}_q \cdot \mathbf{V}, \tag{15.63}$$

then we have

$$\begin{aligned} U(\mathsf{R}) V_q U(\mathsf{R})^\dagger &= \hat{\mathbf{e}}_q \cdot (\mathsf{R}^{-1}\mathbf{V}) = (\mathsf{R}\hat{\mathbf{e}}_q) \cdot \mathbf{V} \\ &= \sum_{q'} V_{q'} D^1_{q'q}(\mathsf{R}), \end{aligned} \tag{15.64}$$

where we use Eq. (15.46).

An important example of a higher order irreducible tensor operator is the electric quadrupole operator, which is a $k = 2$ operator. We will present a separate discussion of this operator at a later point.

By restricting Eq. (15.62) to infinitesimal rotations, it is easy to derive the following commutation relations:

$$\begin{aligned} [J_z, T^k_q] &= \hbar q \, T^k_q \\ [J_\pm, T^k_q] &= \hbar \sqrt{(k \mp q)(k \pm q + 1)} \, T^k_{q\pm1}, \\ \sum_i [J_i, [J_i, T^k_q]] &= \hbar^2 k(k+1) \, T^k_q. \end{aligned} \tag{15.65}$$

Conversely, one can show that Eqs. (15.65) implies Eq. (15.62), by building up finite rotations out of infinitesimal ones. Therefore either Eq. (15.62) or Eqs. (15.65) can be taken as the definition of an irreducible tensor operator.

Now we return to the Wigner-Eckart theorem, which concerns matrix elements of the form $\langle \alpha j m | T^k_q | \alpha' j' m' \rangle$, that is, matrix elements of an irreducible tensor operator with respect to the standard angular momentum basis. First we will present the general idea.

Let us suppress the bra $\langle\alpha jm|$ in the matrix element and consider the ket $T^k_q|\alpha' j' m'\rangle$, or, more exactly, the $(2k+1)(2j'+1)$ kets of this form obtained by letting q and m' vary over their respective ranges. It turns out that multiplying an irreducible tensor operator T^k_q by a ket $|\alpha' j' m'\rangle$ has much in common with forming the tensor product of two kets with the same angular momentum quantum numbers, that is, $|kq\rangle \otimes |j'm'\rangle$. More precisely, it is the custom to think of this tensor product in the reverse order,

$$|j'm'\rangle \otimes |kq\rangle = |j'km'q\rangle, \tag{15.66}$$

where we invoke the notation (14.20) for the kets of the uncoupled basis. In the case of the tensor product of kets, we know that the kets of the uncoupled basis can be expressed as linear combinations of the kets $|jm\rangle$ of the coupled basis, which are eigenkets of the total J^2 and J_3. The expansion is

$$|j'km'q\rangle = \sum_{jm} |jm\rangle\langle jm|j'km'q\rangle, \tag{15.67}$$

in which the expansion coefficients are the Clebsch-Gordan coefficients. Conversely, the kets of the coupled basis (the eigenkets of total J^2 and J_3) can be expressed as linear combinations of the kets of the uncoupled basis,

$$|jm\rangle = \sum_{m'q} |j'km'q\rangle\langle j'km'q|jm\rangle. \tag{15.68}$$

What the tensor product of two kets (15.66) has in common with the product $T^k_q|\alpha' j' m'\rangle$ (an operator times a ket) is that they both have similar transformation properties under rotations. In particular, the ket $T^k_q|\alpha' j' m'\rangle$ can be expressed as a linear combination of eigenkets of J^2 and J_3 with the same expansion coefficients (the Clebsch-Gordan coefficients) seen in Eq. (15.67). This means that when we multiply $T^k_q|\alpha' j' m'\rangle$ on the left by the bra $\langle\alpha jm|$ to obtain the desired matrix element, we will select out a single term in a series which looks just like Eq. (15.67), and that term will be proportional to the Clebsch-Gordan coefficient $\langle jm|j'km'q\rangle$. This Clebsch-Gordan coefficient captures all the dependence of the matrix element $\langle\alpha jm|T^k_q|\alpha' j' m'\rangle$ on the magnetic quantum numbers m, q and m'.

Now we state the Wigner-Eckart theorem, which says that the matrix element $\langle\alpha jm|T^k_q|\alpha' j' m'\rangle$ can be written as the product of the Clebsch-Gordan coefficient $\langle jm|j'km'q\rangle$ times a quantity which is independent of m, q, and m'. We write this in the form,

$$\boxed{\langle\alpha jm|T^k_q|\alpha' j' m'\rangle = \frac{1}{\sqrt{2j+1}}\,\langle\alpha j\|T^k\|\alpha' j'\rangle\,\langle jm|j'km'q\rangle,} \tag{15.69}$$

where the factor $1/\sqrt{2j+1}$ is merely a conventional factor of convenience, and where the quantity $\langle\alpha j\|T^k\|\alpha' j'\rangle$ is called the *reduced matrix element*, and is independent of m, q, and m'. The reduced matrix element depends only on the irreducible tensor operator T^k and on the two irreducible subspaces $\mathcal{E}_{\alpha j}$ and $\mathcal{E}_{\alpha' j'}$ which it links. The conventional factor of $\sqrt{2j+1}$ is unfortunate for pedagogical purposes, because it makes the Wigner-Eckart theorem look more complicated than it really is.

Probably the most useful application of the Wigner-Eckart theorem is that it allows us to easily write down selection rules for the given matrix element, based on the selection rules of the Clebsch-Gordan coefficient occurring in Eq. (15.69). In general, a *selection rule* is a rule which tells us when a matrix element must vanish on account of some symmetry consideration. The Wigner-Eckart theorem provides us with all the selection rules which follow from rotational symmetry; a given matrix element may have other selection rules based on other symmetries (e.g., parity). The selection rules which follow from the Wigner-Eckart theorem are that the matrix element $\langle\alpha jm|T^k_q|\alpha' j'm'\rangle$ vanishes unless $m = m' + q$ and j takes on one of the values, $j = |j'-k|, |j'-k|+1, \ldots, j'+k$.

Furthermore, suppose we actually have to evaluate the matrix elements $\langle\alpha jm|T^k_q|\alpha' j'm'\rangle$ for all $(2k+1)(2j'+1)$ possibilities we get by varying q and m'. We must do this, for example, in computing atomic transition rates. (We need not vary m independently, since the selection rules enforce $m = m' + q$.) Then the Wigner-Eckart theorem tells us that we actually only have to do one of these matrix elements (presumably, whichever is the easiest), and all the rest follow by computing (or looking up) Clebsch-Gordan coefficients.

Now we turn to a proof of the Wigner-Eckart theorem. We begin by considering the $(2k+1)(2j'+1)$ kets $T^k_q|\alpha' j'm'\rangle$. We will show that the space spanned by these kets can be decomposed into irreducible subspaces exactly as when we combine angular momenta according to $j' \otimes k$. To begin, we define new vectors, which, if we were carrying out the analogous problem of combining the angular momenta according to $j' \otimes k$, would give us standard basis kets $|jm\rangle$ in the decomposition of $j' \otimes k$. To this end, we define the ket

$$|X; jm\rangle = \sum_{qm'} T^k_q|\alpha' j'm'\rangle\langle j'km'q|jm\rangle, \tag{15.70}$$

where we follow the pattern of Eq. (15.68) in the sum on the right hand side. The quantum numbers (jm) in the Clebsch-Gordan coefficient on the right hand side are arbitrary (although we are mainly interested in those values which will give nonzero Clebsch-Gordan coefficients), so the sum itself is a ket which depends on (jm), as indicated by the ket on the left hand side. The symbol X stands for all the other parameters on which the sum depends, i.e., $X = (k\alpha' j')$. At this point the symbols (jm) are just labels of the ket on the left hand side, although soon we will prove that $|X; jm\rangle$ actually is an eigenket of J^2 and J_3 with quantum numbers (jm). In fact, we will show more: that the kets $|X; jm\rangle$ for fixed X and j but variable m are linked by raising and lowering operators with standard phase conventions, i.e., the set of $2j+1$ kets obtained by varying m form a standard basis in an irreducible subspace under rotations. (At least, this is true if the kets $|X; jm\rangle$ for $m = -j, \ldots, j$ do not vanish.)

Many books, including Sakurai's, prove these facts by working with the commutation relations (15.65), but I think it is a little easier to work with rotation operators. Our first step is to work out the transformation law of the kets $|X; jm\rangle$ under rotations. The result turns out to be

$$U|X; jm\rangle = \sum_{m'} |X; jm'\rangle \, D^j_{m'm}(U), \tag{15.71}$$

which of course is exactly the transformation law for the standard basis of an irreducible subspace [see Eq. (11.56)]. The proof of Eq. (15.71) is straightforward but involves some lengthy summations, and therefore is placed below in an Appendix.

Now it is a fact that if a set of vectors such as $|X; jm\rangle$ transform as a standard basis of an irreducible subspace, then they are a standard basis in an irreducible subspace (unless they vanish). To show this explicitly, we specialize the rotation operator U in Eq. (15.71) to an infinitesimal rotation as in Eq. (15.14), which we substitute into Eq. (11.46). This gives

$$D^j_{m'm}(U) = \langle jm'|U|jm\rangle = \delta_{mm'} - \frac{i}{\hbar}\theta \langle jm'|\hat{\mathbf{n}}\cdot\mathbf{J}|jm\rangle. \tag{15.72}$$

Then Eq. (15.71) becomes,

$$(\hat{\mathbf{n}}\cdot\mathbf{J})|X; jm\rangle = \sum_{m'} |X; jm'\rangle \, \langle jm'|(\hat{\mathbf{n}}\cdot\mathbf{J})|jm\rangle. \tag{15.73}$$

Now we let $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, or $\hat{\mathbf{z}}$ and we use Eqs. (11.32) and (11.37) for the standard matrix elements of the components of $\mathbf{J}$, to obtain

$$J_z|X; jm\rangle = m\hbar \, |X; jm\rangle, \tag{15.74}$$

$$J_\pm|X; jm\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)} \, |X; j, m \pm 1\rangle, \tag{15.75}$$

from which we easily derive

$$J^2|X; jm\rangle = \hbar^2 j(j+1) \, |X; jm\rangle. \tag{15.76}$$

It is possible that the ket $|X; jm\rangle$ vanishes for some value of m, in which case the raising and lowering equations in Eq. (15.75) show that the kets vanish for all values of m. If

$|X; jm\rangle$ does not vanish for some value of m, then the $2j+1$ vectors obtained by raising and lowering form a standard basis (generally unnormalized) in an irreducible subspace with angular momentum quantum number j. Thus the claims made below Eq. (15.70) are justified.

Now just as we did above in Eq. (15.57), we can argue that $|X; jm\rangle$ must be a linear combination of $|\alpha jm\rangle$ for the given values of (jm) but generally all values of α. That is, we can write

$$|X; jm\rangle = \sum_\alpha C^{kjj'}_{\alpha\alpha'} |\alpha jm\rangle. \tag{15.77}$$

The expansion coefficients depend on the indicated parameters, but not on m as an application of raising or lowering operators will show, following the steps in Eqs. (15.57)–(15.59).

Now we return to Eq. (15.70) and solve for $T^k_q|\alpha' j' m'\rangle$ in terms of the states $|X; jm\rangle$, using the inverse Clebsch-Gordan expansion as in Eq. (15.67). This gives

$$\begin{aligned} T^k_q|\alpha' j' m'\rangle &= \sum_{jm} |X; jm\rangle\langle jm|j'km'q\rangle \\ &= \sum_{\alpha jm} C^{kjj'}_{\alpha\alpha'}\, |\alpha jm\rangle\langle jm|j'km'q\rangle. \end{aligned} \tag{15.78}$$

Finally, we multiply on the left by $\langle\alpha jm|$ to obtain

$$\langle\alpha jm|T^k_q|\alpha' j' m'\rangle = C^{kjj'}_{\alpha\alpha'}\, \langle jm|j'km'q\rangle, \tag{15.79}$$

which gives us the Wigner-Eckart theorem if we set

$$C^{kjj'}_{\alpha\alpha'} = \frac{1}{\sqrt{2j+1}}\, \langle\alpha j\|T^k\|\alpha' j'\rangle. \tag{15.80}$$

This concludes the proof of the Wigner-Eckart theorem.

As we have seen, the idea behind the Wigner-Eckart theorem is that a product of an irreducible tensor operator T^k_q times a ket of the standard basis $|\alpha' j' m'\rangle$ transforms under rotations exactly as the tensor product of two kets of standard bases with the same quantum numbers, $|j'm'\rangle \otimes |kq\rangle$. Similarly, it turns out that the product of two irreducible tensor operators, say, $X^{k_1}_{q_1} Y^{k_2}_{q_2}$, transforms under rotations exactly like the tensor product of kets with the same quantum numbers, $|k_1 q_1\rangle \otimes |k_2 q_2\rangle$. In particular, such a product of operators can be represented as a linear combination of irreducible tensor operators with order k lying in the range $|k_1 - k_2|, \ldots, k_1 + k_2$, with coefficients which are Clebsch-Gordan coefficients. That is, we can write

$$X^{k_1}_{q_1} Y^{k_2}_{q_2} = \sum_{kq} T^k_q\, \langle kq|k_1 k_2 q_1 q_2\rangle, \tag{15.81}$$

where the T^k_q are new irreducible tensor operators. To prove this, we first solve for T^k_q,

$$T^k_q = \sum_{q_1 q_2} X^{k_1}_{q_1} Y^{k_2}_{q_2} \, \langle k_1 k_2 q_1 q_2 | kq \rangle, \tag{15.82}$$

and then prove that T^k_q satisfies the definition (15.62) of an irreducible tensor operator. This latter proof involves some lengthy summations, and is relegated to the Appendix.

Appendix. Proof of Eqs. (15.71) and (15.82).

To prove Eq. (15.71), we apply a rotation operator U to Eq. (15.70), obtaining

$$\begin{aligned} U|X; jm\rangle &= \sum_{qm'} (U\, T^k_q\, U^\dagger)\, U|\alpha' j' m'\rangle \langle j' k m' q | jm\rangle \\ &= \sum_{qm'} \sum_{q'm''} T^k_{q'} |\alpha' j' m''\rangle \langle j' k m' q | jm\rangle \, D^k_{q'q}(U)\, D^{j'}_{m''m'}(U), \end{aligned} \tag{15.83}$$

where we use Eqs. (15.62) and (11.56). Next, for the product of two D-matrices, we use Eq. (14.37), in which we make the following replacements, $(j_1, m_1, m_1') \to (j', m'', m')$, $(j_2, m_2, m_2') \to (k, q', q)$, and $(j, m', m) \to (j'', m_1, m_2)$. Thus we obtain,

$$\begin{aligned} U|X; jm\rangle &= \sum_{qm'} \sum_{q'm''} \sum_{j''m_1} \sum_{m_2} T^k_{q'} |\alpha' j' m''\rangle \langle j' k m' q | jm\rangle \\ &\qquad \times \langle j' k m'' q' | j'' m_2\rangle \langle j'' m_1 | j' k m' q\rangle \, D^{j''}_{m_2 m_1}(U) \\ &= \sum_{q'm''} \sum_{j''m_1} \sum_{m_2} T^k_{q'} |\alpha' j' m''\rangle \langle j' k m'' q' | j'' m_2\rangle \, \delta_{jj''}\, \delta_{mm_1} D^{j''}_{m_2 m_1}(U) \\ &= \sum_{qm'} \sum_{m''} T^k_q |\alpha' j' m'\rangle \langle j' k m' q | jm''\rangle D^j_{m''m}(U) \\ &= \sum_{m''} |X; jm''\rangle \, D^j_{m''m}(U). \end{aligned} \tag{15.84}$$

In the second equality we use the orthonormality relation (14.30a) to combine the first and third Clebsch-Gordan coefficients of the long sum into Kronecker deltas, which then allow the j'' and m_1 sums to be done. On the third equality we also rearrange indices, making the replacements $(q', m'', m_2) \to (q, m', m'')$. The final equality follows from the definition of the vectors $|X; jm\rangle$ in Eq. (15.70). Thus Eq (15.71) is proven.

To prove Eq. (15.82), we conjugate both sides with a rotation operator, obtaining,

$$UT^k_q U^\dagger = \sum_{q_1 q_2} (U X^{k_1}_{q_1} U^\dagger)(U Y^{k_2}_{q_2} U^\dagger) \, \langle k_1 k_2 q_1 q_2 | kq\rangle$$

$$
\begin{aligned}
&= \sum_{q_1 q_2} \sum_{q_1' q_2'} X^{k_1}_{q_1'} Y^{k_2}_{q_2'} \, D^{k_1}_{q_1' q_1} D^{k_2}_{q_2' q_2} \, \langle k_1 k_2 q_1 q_2 | kq \rangle \\
&= \sum_{q_1 q_2} \sum_{q_1' q_2'} \sum_{k_3 q_3 q_3'} X^{k_1}_{q_1'} Y^{k_2}_{q_2'} \, \langle k_1 k_2 q_1' q_2' | k_3 q_3' \rangle \\
&\qquad \times D^{k_3}_{q_3' q_3} \, \langle k_3 q_3 | k_1 k_2 q_1 q_2 \rangle \langle k_1 k_2 q_1 q_2 | kq \rangle \\
&= \sum_{q_1' q_2'} \sum_{k_3 q_3 q_3'} X^{k_1}_{q_1'} Y^{k_2}_{q_2'} \, \langle k_1 k_2 q_1' q_2' | k_3 q_3' \rangle \, D^{k_3}_{q_3' q_3} \, \delta_{k k_3} \, \delta_{q q_3} \\
&= \sum_{q_1' q_2' q_3'} X^{k_1}_{q_1'} Y^{k_2}_{q_2'} \, \langle k_1 k_2 q_1' q_2' | k q_3' \rangle \, D^{k}_{q_3' q} \\
&= \sum_{q_3'} T^{k}_{q_3'} \, D^{k}_{q_3' q}, \qquad (15.85)
\end{aligned}
$$

where we use Eq. (15.62) in the second equality, Eq. (14.37) in the third [along with the change of indices, $(j_1 m_1 m_1') \to (k_1 q_1' q_1)$, $(j_2 m_2 m_2') \to (k_2 q_2' q_2)$ and $(jmm') \to (k_3 q_3' q_3)$], and Eq. (14.30a) in the fourth. Thus, T^k_q does transform as an irreducible tensor operator.