Physics 209

Fall 2002

Homework 9

Due Saturday, November 1 at 5:00pm.

Note: Due to the exam in 221A, this homework is due on Saturday, November 1 instead of Friday, October 31. However, the building is only open during limited hours on Saturday. If you do not have a key, you will have to make arrangements to have your homework delivered on time, or else turn it in on Friday.

1. In class we derived the $3+1$ version of the relativistic Lagrangian for a particle in a given electric and magnetic field,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = -mc^2\sqrt{1 - |\dot{\mathbf{x}}|^2/c^2} - e\Phi + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}. \tag{1}$$

(a) Show that the Euler-Lagrange equations give the correct equations of motion.

(b) Show that this Lagrangian is regular (you can solve for $\dot{\mathbf{x}}$ in terms of $\mathbf{p}$), and find the Hamiltonian $H(\mathbf{x}, \mathbf{p})$

(c) Show that Hamilton's equations give the correct equations of motion.

2. Noether's theorem for Lagrangians $L(q, \dot{q})$ was described in class as follows. We let $q \to Q(q, \epsilon)$ be a mapping of configuration space onto itself. We say that this mapping is a *symmetry* of the Lagrangian if

$$L\Big(Q(q,\epsilon), \frac{dQ(q,\epsilon)}{dt}\Big) = L(q, \dot{q}) + \frac{d\Lambda(q,\epsilon)}{dt} \tag{2}$$

for some function Λ, where q, Q etc are abbreviations for multicomponent coordinate vectors, q_i, Q_i, etc. Also, in Eq. (2) dQ/dt and $d\Lambda/dt$ are abbreviations for

$$\frac{dQ_i(q,\epsilon)}{dt} = \sum_j \frac{\partial Q_i}{\partial q_j}\dot{q}_j \tag{3}$$

and

$$\frac{d\Lambda(q,\epsilon)}{dt} = \sum_i \frac{\partial \Lambda}{\partial q_i}\dot{q}_i, \tag{4}$$

so that the Lagrangian before and after the transformation is a function of q and $\dot{q}$. Equation (2) means that the equations of motion are the same before and after the symmetry transformation.

As explained in class, we only use this symmetry property for infinitesimal ϵ, which means we expand out to first order in ϵ to obtain an identity involving the Lagrangian, or else we differentiate Eq. (2) with respect to ϵ and set $\epsilon = 0$, which is the same thing. This gives the identity,

$$\sum_i \Big(\frac{\partial L}{\partial q_i} F_i + \frac{\partial L}{\partial \dot{q}_i} \frac{dF_i}{dt} \Big) = \frac{dG}{dt}, \tag{5}$$

where

$$F_i(q) = \frac{\partial Q_i(q,\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0}, \tag{6}$$

and

$$G(q) = \frac{\partial \Lambda(q,\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0}. \tag{7}$$

Equation (5) is just an identity satisfied by the Lagrangian, due to the symmetry. But now we use the Euler-Lagrange equations,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) = \frac{\partial L}{\partial q_i}, \tag{8}$$

to write Eq. (5) in the form,

$$\frac{d}{dt}\Big(\sum_i F_i p_i - G\Big) = 0. \tag{9}$$

This gives us the conserved quantity corresponding to the symmetry.

(a) Consider the relativistic motion of a charged particle of charge e in a constant, uniform magnetic field, $\mathbf{B} = B_0 \hat{\mathbf{z}}$. Use the covariant Lagrangian,

$$L\Big(x^\mu, \frac{dx^\mu}{d\sigma}\Big) = mc\sqrt{\frac{dx^\mu}{d\sigma}\frac{dx_\mu}{d\sigma}} - \frac{eB_0}{c} x \frac{dy}{d\sigma}, \tag{10}$$

where as explained in class t is regarded as a 4th q and σ is an arbitrary parameter of the orbits. Equation (10) effectively uses the vector potential, $\mathbf{A} = B_0 x \hat{\mathbf{y}}$. Find the constants of motion associated with translations in the x, y, z and t directions. Find the constant of motion associated with rotations in the x-y plane. Express your constants of motion in terms of the kinetic (not canonical) momentum, or equivalently in terms of the world velocity. Show that the constants associated with x and y displacements are closely related to the guiding center position.

Notational suggestion: Since the covariant, kinetic 4-momentum p_μ has components,

$$p_\mu = \begin{pmatrix} E/c \\ -\mathbf{p} \end{pmatrix}, \tag{11}$$

use the notation p_1, p_2, p_3 for p_μ with $\mu = 1, 2, 3$, and p_x, p_y, p_z for the usual components of $\mathbf{p}$, so that $p_x = -p_1$, etc.

(b) This is probably the hardest part of this homework, so you might want to do it last. Consider the motion of a nonrelativistic charged particle of charge e in the field of a magnetic monopole,

$$\mathbf{B}(\mathbf{x}) = g\frac{\mathbf{x}}{r^3}, \tag{12}$$

where g is the strength of the monopole and $r = |\mathbf{x}|$. Assume the monopole is infinitely massive and located at the origin. The nonrelativistic Lagrangian for this system is

$$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}), \tag{13}$$

where $\mathbf{B} = \nabla\times\mathbf{A}$. It is possible to work out explicit formulas for $\mathbf{A}$ which produce the magnetic field (12), but the formulas are ugly, so we will not attempt to do that. Instead, we will just use the property $\mathbf{B} = \nabla\times\mathbf{A}$, with $\mathbf{B}$ given by Eq. (12).

Consider rotations about the axis $\hat{\mathbf{n}}$, so that for small angles of rotation θ, $\mathbf{x}$ is replaced by $\mathbf{x} + \theta(\hat{\mathbf{n}}\times\mathbf{x})$. Show that such rotations are a symmetries of the Lagrangian (13), and find the corresponding vector of conserved quantities (a vector since $\hat{\mathbf{n}}$ is arbitrary). Hint: the conserved quantities will be gauge-invariant, that is, you should be able to find an expression for them that does not involve $\mathbf{A}$.

3. Fermat's principle in optics says that light rays $\mathbf{x}(\lambda)$ are stationary paths of the following functional, the "optical path,"

$$P[\mathbf{x}(\lambda)] = c\int dt = \int n(\mathbf{x})\,ds = \int_{\lambda_0}^{\lambda_1} n(\mathbf{x})\left|\frac{d\mathbf{x}}{d\lambda}\right|\,d\lambda, \tag{14}$$

where $n(\mathbf{x})$ is the index of refraction, $\mathbf{x} = (x, y, z)$, and λ is an arbitrary parameter.

Suppose the medium is one in which $n(\mathbf{x})$ only depends on z, so that the system has translational symmetry in the x- and y-directions. For example, we might have a discontinuous change from one medium to another, with the interface being at $z = \text{const}$. Show that Snell's law is a consequence of Noether's theorem.

4. In class we modelled the transverse vibrations of a string as a discrete set of N masses m spaced at intervals of Δx with Lagrangian,

$$L = \sum_{i=1}^{N} \frac{m}{2}\dot{y}_i^2 - \sum_{i=1}^{N+1} \frac{k}{2}(y_i - y_{i-1})^2, \tag{15}$$

where k is the spring constant for the spring connecting mass i and mass $i-1$. All springs have the same k. Then we took the limit in which $N \to \infty$, $m = \rho\Delta x$, $k = \kappa/\Delta x$, where ρ is the mass per unit length of the string and κ is the spring constant of the whole string times the length of the string. This gave the field Lagrangian,

$$L = \int dx\, \mathcal{L}, \tag{16}$$

where the Lagrangian density is

$$\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{\kappa}{2}\left(\frac{\partial y}{\partial x}\right)^2. \tag{17}$$

Construct the classical Hamiltonian H for the discretized version of the vibrating string. Take the limit $N \to \infty$ and show that H can be written as the x integral of a Hamiltonian density $\mathcal{H}$, and find an expression for $\mathcal{H}$. The Hamiltonian density $\mathcal{H}$ is interpreted as the energy density of the system.

Show that there exists a function S, depending on $y(x,t)$ and/or its various derivatives, such that

$$\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial S}{\partial x} = 0. \tag{18}$$

This is a one-dimensional continuity equation, and S is interpreted as the energy flux of the system (like the Poynting vector in electromagnetism).